Weakly Useful Sequences

Stephen A. Fenner[1*] and Jack H. Lutz[2**] and Elvira Mayordomo[3***]

[1] University of Southern Maine, Portland, Maine, USA. E-mail: fenner@usm.maine.edu.
[2] Iowa State University, Ames, Iowa, USA. E-mail: lutz@iastate.edu.
[3] University of Zaragoza, Zaragoza, SPAIN. E-mail: emayordomo@mcps.unizar.es.

Abstract. An infinite binary sequence x is defined to be

1. *strongly useful* if there is a recursive time bound within which *every* recursive sequence is Turing reducible to x; and
2. *weakly useful* if there is a recursive time bound within which all the sequences in a non-measure 0 subset of the set of recursive sequences are Turing reducible to x.

Juedes, Lathrop, and Lutz (1994) proved that every weakly useful sequence is strongly deep in the sense of Bennett (1988) and asked whether there are sequences that are weakly useful but not strongly useful.
The present paper answers this question affirmatively. The proof is a direct construction that combines the recent *martingale diagonalization* technique of Lutz (1994) with a new technique, namely, the construction of a sequence that is "recursively deep" with respect to an arbitrary, given uniform reducibility. The *abundance* of such recursively deep sequences is also proven and used to show that every weakly useful sequence is recursively deep with respect to every uniform reducibility.

1 Introduction

It is a truism that the usefulness of a data object does not vary directly with its information content. For example, consider two infinite binary strings, χ_K, the characteristic sequence of the halting problem (whose nth bit is 1 if and only if the nth Turing machine halts on input n), and z, a sequence that is algorithmically random in the sense of Martin-Löf [10]. The following facts are well-known.

1. The first n bits of χ_K can be specified using only $O(\log n)$ bits of information, namely, the *number* of 1's in the first n bits of χ_K [2].

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2. The first n bits of z cannot be specified using significantly fewer than n bits of information [10].
3. Oracle access to χ_K would enable one to decide *any* recursive sequence in polynomial time (i.e., decide the nth bit of the sequence in time polynomial in the length of the binary representation of n) [11].
4. Even with oracle access to z, most recursive sequences cannot be computed in polynomial time. (This appears to be folklore, known at least since [3].)

Facts (i) and (ii) tell us that χ_K contains far less information than z. In contrast, facts (iii) and (iv) tell us that χ_K is computationally much more useful than z. That is, the information in χ_K is "more usefully organized" than that in z.

Bennett [3] introduced the notion of *computational depth* (also called "logical depth") in order to quantify the degree to which the information in an object has been organized. In particular, for infinite binary sequences, Bennett defined two "levels" of depth, *strong depth* and *weak depth*, and argued that the above situation arises from the fact that χ_K is strongly deep, while z is not even weakly deep. (The present paper is motivated by the study of computational depth, but does not directly use strong or weak depth, so definitions are omitted here. The interested reader is referred to [3], [7], or [6] for details, and for related aspects of algorithmic information theory.)

Investigating this matter further, Juedes, Lathrop, and Lutz [6] considered two "levels of usefulness" for infinite binary sequences. Specifically, let $\{0,1\}^\infty$ be the set of all infinite binary sequences, let REC be the set of all recursive elements of $\{0,1\}^\infty$, and, for $x \in \{0,1\}^\infty$ and $t\colon \mathbf{N} \to \mathbf{N}$, let $\mathrm{DTIME}^x(t)$ be the set of all $y \in \{0,1\}^\infty$ for which there exists an oracle Turing machine M that, on input $n \in \mathbf{N}$ with oracle x, computes $y[n]$, the nth bit of y, in at most $t(\ell)$ steps, where ℓ is the number of bits in the binary representation of n. Then a sequence $x \in \{0,1\}^\infty$ is defined to be *strongly useful* if there is a recursive time bound $t\colon \mathbf{N} \to \mathbf{N}$ such that $\mathrm{DTIME}^x(t)$ contains all of REC. A sequence $x \in \{0,1\}^\infty$ is defined to be *weakly useful* if there is a recursive time bound $t\colon \mathbf{N} \to \mathbf{N}$ such that $\mathrm{DTIME}^x(t)$ contains a non-measure 0 subset of REC, in the sense of resource-bounded measure [9]. That is, x is weakly useful if access to x enables one to decide a *nonnegligible set* of recursive sequences within some fixed recursive time bound. No recursive or algorithmically random sequence can be weakly useful. It is evident that χ_K is strongly useful, and that every strongly useful sequence is weakly useful.

Juedes, Lathrop, and Lutz [6] generalized Bennett's result that χ_K is strongly deep by proving that *every* weakly useful sequence is strongly deep. This confirmed Bennett's intuitive arguments by establishing a definite relationship between computational depth and computational usefulness. It also substantially extended Bennett's result on χ_K by implying (in combination with known results of recursion theory [10, 13, 4, 5]) that *all* high Turing degrees and *some* low Turing degrees contain strongly deep sequences.

Notwithstanding this progress, Juedes, Lathrop, and Lutz [6] left a critical question open: Do there exist weakly useful sequences that are not strongly useful? The main result of the present paper, proved in Section 4, answers this question affirmatively. This establishes the existence of strongly deep sequences that are not strongly useful. More importantly, it indicates a need for further investigation of the class of weakly useful sequences.

The proof of our main result is a direct construction that combines the *martingale diagonalization* technique recently introduced by Lutz [8] with a new technique, namely, the construction of a sequence that is *recursively F-deep*, where F is an arbitrary uniform reducibility. This notion of uniform recursive depth, defined and investigated in Section 3, is closely related to Bennett's notion of weak depth.

In addition to using specific constructions of recursively F-deep sequences, we prove that, for each uniform reducibility F, *almost every* sequence in REC is recursively F-deep. This implies that every weakly useful sequence is, for every uniform reducibility F, recursively F-deep.

2 Preliminaries

We use $\mathbf{N}$ to denote the set of natural numbers (including 0), and we use $\mathbf{Q}$ to denote the set of rational numbers. We write $[\![\varphi]\!]$ for the Boolean value of a condition φ, i.e.,

$$[\![\varphi]\!] = \textbf{if } \varphi \textbf{ then } 1 \textbf{ else } 0.$$

Throughout this paper, we identify each set $A \subseteq \mathbf{N}$ with its characteristic sequence $\chi_A \in \{0,1\}^\infty$, whose nth bit is $\chi_A[n] = [\![n \in A]\!]$. For any $x, y \in \{0,1\}^* \cup \{0,1\}^\infty$, we write $x \sqsubseteq y$ to mean that x is a prefix of y, and if in addition, $x \neq y$, we may write $x \sqsubset y$.

We fix a recursive bijection $\langle \cdot, \cdot \rangle : \mathbf{N}^2 \to \mathbf{N}$, monotone in both arguments, such that $i \leq \langle i, j \rangle$ and $j \leq \langle i, j \rangle$ for all $i, j \in \mathbf{N}$.

In the proof of Theorem 12, we will deal extensively with *partial characteristic functions*, i.e., functions with domain a subset of $\mathbf{N}$ and with range $\{0,1\}$. We will identify binary strings with characteristic functions whose domains are finite initial segments of $\mathbf{N}$. If σ and τ are partial characteristic functions, we let $\mathrm{dom}(\sigma)$ denote the domain of σ, and say that σ and τ are *compatible* if they agree on all elements in $\mathrm{dom}(\sigma) \cap \mathrm{dom}(\tau)$. We say that σ *is extended by* τ ($\sigma \sqsubseteq \tau$) if σ and τ are compatible and $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)$ (if in addition $\sigma \neq \tau$, we write $\sigma \sqsubset \tau$). If σ and τ are compatible, we let $\sigma \cup \tau$ be their smallest common extension.

We will often think of $\mathbf{N}$ being split up into *columns* $0, 1, 2, \ldots$ where the ith column is $\{\langle i, j \rangle \mid j \in \mathbf{N}\}$. If $A \subseteq \mathbf{N}$, then the *ith strand of A* is defined as $A_i = \{x \mid \langle i, x \rangle \in A\}$. If σ is a partial characteristic function and $n \in \mathbf{N}$, then $\sigma[< n]$ denotes σ restricted to the domain $\{0, \ldots, n-1\}$, and $\sigma[i, < n]$ denotes the unique partial characteristic function τ such that for all x,

$$\tau(x) = \begin{cases} \sigma(\langle i, x \rangle) & \text{if } x < n \text{ and } \sigma(\langle i, x \rangle) \text{ is defined,} \\ \text{undefined otherwise.} \end{cases}$$

That is, $\sigma[i, < n]$ results from "excising" the first n bits of σ from the ith column. Inversely, if w is a binary string, then $\{i\} \times w$ denotes the unique partial characteristic function τ such that $\tau(\langle i, x\rangle) = w(x)$ for all $x < |w|$, and is undefined on all other arguments. That is, $\{i\} \times w$ is w "translated" over to the ith column. Of particular importance will be the finite characteristic function defined for an arbitrary $C \subseteq \mathbf{N}$ and $k, y \in \mathbf{N}$ as

$$\xi^C(k, y) = \bigcup_{k' < k} \{k'\} \times C[k', < y].$$

In other words, $\xi^C(k, y)$ is χ_C restricted to the "rectangle" of width k and height y, with a corner at the origin.

Weakly useful sequences are defined (in Section 1) in terms of *recursive measure*, a special case of the resource-bounded measure developed by Lutz [9]. We very briefly sketch the elements of this theory, referring the reader to [9, 8] for motivation, details, and intuition.

Definition 1. A *martingale* is a function $d : \{0,1\}^* \to [0, \infty)$ such that, for all $w \in \{0,1\}^*$, $d(w) = \frac{d(w0)+d(w1)}{2}$.

Definition 2. A martingale d is *recursive* if there is a total recursive function $\hat{d} : \mathbf{N} \times \{0,1\}^* \to \mathbf{Q}$ such that, for all $r \in \mathbf{N}$ and $w \in \{0,1\}^*$,

$$\left|\hat{d}(r, w) - d(w)\right| \leq 2^{-r}.$$

Definition 3. A martingale d *succeeds* on a sequence $x \in \{0,1\}^\infty$ if

$$\limsup_{n\to\infty} d(x[0 \ldots n-1]) = \infty,$$

where $x[0 \ldots n-1]$ is the n-bit prefix of x. The *success set* of a martingale d is

$$S^\infty[d] = \{x \in \{0,1\}^\infty \mid d \text{ succeeds on } x\}.$$

Definition 4. Let $X \subseteq \{0,1\}^\infty$.

1. X has *recursive measure 0*, and we write $\mu_{\mathrm{rec}}(X) = 0$, if there is a recursive martingale d such that $X \subseteq S^\infty[d]$.
2. X has *recursive measure 1*, and we write $\mu_{\mathrm{rec}}(X) = 1$, if $\mu_{\mathrm{rec}}(X^c) = 0$, where $X^c = \{0,1\}^\infty - X$ is the complement of X.
3. X has *measure 0 in* REC, and we write $\mu(X \mid \mathrm{REC}) = 0$, if $\mu_{\mathrm{rec}}(X \cap \mathrm{REC}) = 0$.
4. X has *measure 1 in* REC, and we write $\mu(X \mid \mathrm{REC}) = 1$, if $\mu(X^c \mid \mathrm{REC}) = 0$. In this case, we say that X contains *almost every* element of REC.

3 Uniform Recursive Depth

In this section we prove that, for every uniform reducibility F, almost every recursive subset of $\mathbf{N}$ has a certain "depth" property with respect to F. This depth property is used in the proof of our main result in Section 4. It is also of independent interest because it is closely related to Bennett's notion of weak depth [3].

We first make our terminology precise. As in [12], we define a *truth-table condition* (briefly, a *tt-condition*) to be an ordered pair $\tau = ((n_1, \ldots, n_k), g)$, where $k, n_1, \ldots, n_k \in \mathbf{N}$ and $g: \{0,1\}^k \to \{0,1\}$. We write TTC for the class of all tt-conditions. The *tt-value* of a set $B \subseteq \mathbf{N}$ under a tt-condition $\tau = ((n_1, \ldots, n_k), g)$ is the bit

$$\tau^B = g([\![n_1 \in B]\!] \cdots [\![n_k \in B]\!]).$$

A *truth-table reduction* (briefly, a *tt-reduction*) is a total recursive function $F: \mathbf{N} \to \text{TTC}$. If F is a tt-reduction and $F(x) = ((n_1, \ldots, n_k), g)$, then we call $n_1, \ldots, n_k$ the *queries* made by F on input x. A truth-table reduction F naturally *induces* a function $\widehat{F}: \mathcal{P}(\mathbf{N}) \to \mathcal{P}(\mathbf{N})$ defined by

$$\widehat{F}(B) = \{n \in \mathbf{N} \mid F(n)^B = 1\}.$$

In general, we identify a truth-table reduction F with the induced function $\widehat{F}$, writing F for either function and relying on context to avoid confusion.

The following terminology is convenient for our purposes.

Definition 5. A *uniform reducibility* is a total recursive function $F: \mathbf{N} \times \mathbf{N} \to$ TTC.

If F is a uniform reducibility, then we use the notation $F_k(n) = F(k,n)$ for all $k, n \in \mathbf{N}$. We thus regard a uniform reducibility as a recursive sequence $F_0, F_1, F_2, \ldots$ of tt-reductions.

Definition 6. If F is a uniform reducibility and $A, B \subseteq \mathbf{N}$, then we say that A is *F-reducible* to B, and we write $A \leq_F B$, if there exists $k \in \mathbf{N}$ such that $A = F_k(B)$.

Example 1. Fix a recursive time bound, i.e., a total recursive function $t: \mathbf{N} \to \mathbf{N}$. It is routine to construct a uniform reducibility F such that, for all $A, B \subseteq \mathbf{N}$,

$$A \leq_F B \iff A \in \text{DTIME}^B(t).$$

Definition 7. Let F be a uniform reducibility. The *upper F-span* of a set $A \subseteq \mathbf{N}$ is the set

$$F^{-1}(A) = \{B \subseteq \mathbf{N} \mid A \leq_F B\}.$$

Definition 8. Let F be a uniform reducibility. A set $A \subseteq \mathbf{N}$ is *recursively F-deep* if $\mu_{\text{rec}}(F^{-1}(A)) = 0$.

Bennett [3] defines a set $A \subseteq \mathbf{N}$ to be *weakly deep* if A is not tt-reducible to any algorithmically random set B. The above definition is similar in spirit, but it (i) replaces "tt-reducible" with "F-reducible," and (ii) replaces "any algorithmically random set B" with "any set B outside a set of recursive measure 0."

Definition 9. A set $A \subseteq \mathbf{N}$ is *recursively weakly deep* if, for every uniform reducibility F, A is recursively F-deep.

It is easy to see that every recursively weakly deep set is weakly deep.

We now prove the main result of this section. Recalling our identification of subsets of $\mathbf{N}$ with their characteristic sequences, we state this result in terms of sequences but, for convenience, prove it in terms of sets.

Theorem 10. *If F is a uniform reducibility, then almost every sequence in* REC *is recursively F-deep.*

Proof sketch Assume the hypothesis. For each $k, n \in \mathbf{N}$ and $A \subseteq \mathbf{N}$, define the set

$$\mathcal{E}^A_{k,n} = \{B \subseteq \mathbf{N} \mid (\forall 0 \leq m < n)[\![m \in A]\!] = [\![m \in F_k(B)]\!]\}.$$

This is the set of all B such that $F_k(B)$ agrees with A on $0, 1, \ldots, n-1$. In particular,

$$F^{-1}(A) = \bigcup_{k=0}^{\infty} \bigcap_{n=0}^{\infty} \mathcal{E}^A_{k,n}.$$

We regard $\mathcal{E}^A_{k,n}$ as an event in the sample space $\mathcal{P}(\mathbf{N})$ with the uniform distribution. Thus we write $\Pr(\mathcal{E}^A_{k,n})$ for the probability that $B \in \mathcal{E}^A_{k,n}$, where the set $B \subseteq \mathbf{N}$ is chosen probabilistically according to a random experiment in which an independent toss of a fair coin is used to decide membership of each natural number in B.

For each $A \subseteq \mathbf{N}$, define a function $d^A \colon \{0,1\}^* \to [0, \infty)$ by

$$d^A(w) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} 2^{-\frac{k+n}{4}} d^A_{k,n}(w),$$

where, for all $k, n \in \mathbf{N}$ and $w \in \{0,1\}^*$,

$$d^A_{k,n}(w) = \begin{cases} 2^{|w|} \Pr(\mathbf{C}_w \mid \mathcal{E}^A_{k,n}) & \text{if } \Pr(\mathcal{E}^A_{k,n}) > 0, \\ 1 & \text{if } \Pr(\mathcal{E}^A_{k,n}) = 0, \end{cases}$$

where $\mathbf{C}_w = \{A \subset \mathbf{N} \mid w \sqsubset \chi_A\}$. It is routine to check that each d^A is a martingale that is recursive in A.

For each $k, n \in \mathbf{N}$ and $A \subseteq \mathbf{N}$, let

$$N_A(k, n) = \left| \left\{ m \mid 0 \leq m < n \text{ and } \Pr(\mathcal{E}^A_{k,m+1}) \leq \tfrac{1}{2} \Pr(\mathcal{E}^A_{k,m}) \right\} \right|,$$

and let

$$X = \Big\{ A \subseteq \mathbf{N} \mid (\forall k \in \mathbf{N})(\forall^\infty n \in \mathbf{N}) N_A(k,n) > \frac{n}{4} \Big\},$$

where the quantifier $(\forall^\infty n \in \mathbf{N})$ means "for all but finitely many $n \in \mathbf{N}$."

We use the following four claims (proofs are omitted).

Claim 1 *For all $k, n \in \mathbf{N}$ and $A \subseteq \mathbf{N}$,*

$$\Pr(\mathcal{E}^A_{k,n}) \leq 2^{-N_A(k,n)}.$$

Claim 2 *For all $k, n \in \mathbf{N}$ and $A, B \subseteq \mathbf{N}$ satisfying $A = F_k(B)$,*

$$\liminf_{\ell \to \infty} d^A_{k,n}(\chi_B[0 \ldots \ell - 1]) \geq 2^{N_A(k,n)}.$$

Claim 3 *For all $A \in X$, $F^{-1}(A) \subseteq S^\infty[d^A]$.*

Claim 4 $\mu_{\text{rec}}(X) = 1$.

Let

$$\mathcal{D} = \{A \subseteq \mathbf{N} \mid A \text{ is recursively } F\text{-deep}\}.$$

By Claim 3 and the fact that d^A is recursive in A, we must have $X \cap \text{REC} \subseteq \mathcal{D}$. It follows that $\mathcal{D}^c \cap \text{REC} \subseteq X^c$. Claim 4 tells us that $\mu_{\text{rec}}(X^c) = 0$, and hence

$$\mu(\mathcal{D}^c \mid \text{REC}) = \mu_{\text{rec}}(\mathcal{D}^c \cap \text{REC}) = 0,$$

since any subset of a rec-measure 0 set has rec-measure 0. We thus get $\mu(\mathcal{D} \mid \text{REC}) = 1$, which proves the theorem. □ **Theorem 10**

Theorem 11. *Every weakly useful sequence is recursively weakly deep.*

Proof. Assume that A is weakly useful and fix a uniform reducibility F. It suffices to show that $\mu_{\text{rec}}(F^{-1}(A)) = 0$.

Fix a recursive time bound $t: \mathbf{N} \to \mathbf{N}$ such that $\mu(\text{DTIME}^A(t) \mid \text{REC}) \neq 0$. Then there is a uniform reducibility $\widetilde{F}$ such that, for all $B, C, D \subseteq \mathbf{N}$,

$$[\, B \in \text{DTIME}^C(t) \text{ and } C \leq_F D \,] \Longrightarrow B \leq_{\widetilde{F}} D.$$

Let X be the set of all sets that are recursively $\widetilde{F}$-deep. By Theorem 10, $\mu(X \mid \text{REC}) = 1$, so there is a set $B \in X \cap \text{DTIME}^A(t) \cap \text{REC}$. Now $\mu_{\text{rec}}(\widetilde{F}^{-1}(B)) = 0$ (because $B \in X$) and $F^{-1}(A) \subseteq \widetilde{F}^{-1}(B)$ (because $B \in \text{DTIME}^A(t)$), so $\mu_{\text{rec}}(F^{-1}(A)) = 0$. □

4 Main Result

In this section, we prove the existence of weakly useful sequences that are not strongly useful. Our construction uses recursively F-deep sets (for an infinite, nonuniform collection of uniform reducibilities F), but those sets are constructed in a canonical way.

Theorem 12. *There is a sequence that is weakly useful but not strongly useful.*

We include a sketch of the proof of Theorem 12. The proof uses the next proposition, which is of independent interest.

Proposition 13. *If F is a uniform reducibility, then there is a* canonical *recursive, recursively F-deep set, i.e., a set A such that*

$$\mu_{\mathrm{rec}}(\{B \mid (\exists i) A = F_i(B)\}) = 0,$$

and such that for all $x, i \in \mathbf{N}$, $\Pr_C[F_i(C)[i, < x] = A[i, < x]] \leq 2^{-x}$.

We call A above the *canonical recursively F-deep set.*

Proof sketch of Theorem 12 Our proof is an adaptation of the martingale diagonalization method introduced by Lutz in [8]. We will define H one strand at a time to satisfy the following conditions, where $H_0, H_1, H_2, \ldots$ are the strands of H:

1. Each strand H_k is recursive (although H itself cannot be recursive).
2. If d is any recursive martingale, then there is some k such that d fails on H_k.
3. For every recursive time bound t, there is a recursive set A such that $A \notin \mathrm{DTIME}^H(t)$.

These three conditions suffice for our purposes. By Condition 1, the set $J = \{H_0, H_1, H_2, \ldots\} \subseteq \mathrm{REC}$, and by Condition 2, no recursive martingale can succeed on all its elements. Thus $\mu_{\mathrm{rec}}(J) \neq 0$, which makes H weakly useful, since $J \subseteq \mathrm{DTIME}^H(\mathrm{linear})$. Condition 3 ensures that H is not strongly useful.

Fix an arbitrary enumeration $\{t_k\}_{k \in \mathbf{N}}$ of all recursive time bounds, and an enumeration $\{\tilde{d}_k\}_{k \in \mathbf{N}}$ of all recursive martingales. These enumerations need not be uniform in any sense, since at present we are not trying to control the complexity of H. We will define (in order) a number of different objects for each k:

- a uniform reducibility F^k corresponding to t_k.
- a recursive A^k such that $A^k \notin \mathrm{DTIME}^H(t_k)$ (A^k will be the canonical recursively F^k-deep set (cf. Proposition 13,
- a partial characteristic function α_k of finite domain, compatible with all the previous strands of H (ultimately, $\alpha_k \sqsubset H$ for all k),
- martingales $d^{i,j}_{k;q}$ (uniformly recursive over j and q) for all $i, j, q \in \mathbf{N}$ with $i \leq k$, which, taken together, witness that each A^i is recursively F^i-deep, and

– the strand H_k itself, which is designed to make the martingale

$$d'_k = \tilde{d}_k + \sum_{i=0}^{k} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} d^{i,j}_{k;q} \cdot 2^{-q-j}$$

fail on H_k, thus satisfying Condition 2 above. H_k will also participate in a fixed finite number of diagonalizations against tt-reductions from the A^i to H for $i \leq k$.

Fix $k \in \mathbf{N}$, and assume that all the above objects have been defined for all $k' < k$ (define $\alpha_{-1} = \lambda$). Also assume that for each $k' < k$ we have at our disposal programs to compute (uniformly over j and q) $F^{k'}_j$, $A^{k'}$, $H_{k'}$, and $d^{i,j}_{k';q}$ for all $i \leq k'$. Let $\{M_{j,k}\}_{j \in \mathbf{N}}$ be a recursive enumeration of all oracle Turing machines running in time t_k, and for all j let $M'_{j,k}$ be the same as $M_{j,k}$ except that when $M_{j,k}$ makes a query of the form $\langle x, y \rangle$ for $x < k$, $M'_{j,k}$ instead simulates the answer by computing $H_x(y)$ directly. We let F^k_j be the tt-reduction corresponding to $M'_{j,k}$. Note that on any input, F^k_j only makes queries of the form $\langle x, y \rangle$ for $x \geq k$.

We define A^k to be the canonical recursively F^k-deep set constructed in the proof of Proposition 13, therefore,

Fact 1 *For all $j, k, x \in \mathbf{N}$,* $\Pr_C \left[F^k_j(C)[j, < x] = A^k[j, < x] \right] \leq 2^{-x}$.

Let $H_{<k}$ denote the partial characteristic function that agrees with H on all $\langle x, y \rangle$ with $x < k$, and is undefined otherwise. Given α_{k-1}, which is compatible with $H_{<k}$, we define α_k as follows: let $\langle i, j \rangle = k$. If there is a set $C \sqsupset H_{<k} \cup \alpha_{k-1}$ such that $A^i \neq F^i_j(C)$, then we *diagonalize* against F^i_j by letting α_k be the least finite characteristic function extending α_{k-1} that preserves such a miscomputation, i.e., for some C and x such that $A^i(x) \neq F^i_j(x)^C$, α_k will agree with C on all queries made by F^i_j on input x. If no such C exists, let $\alpha_k = \alpha_{k-1}$.

Now fix any i and j with $i \leq k$. We would like to define a martingale that succeeds on all B such that $A^i = F^i_j(B)$. We cannot do this directly, because any given tt-reduction F^i_j from A^i to H might make queries on many different columns at once, and our martingales can only act on one column at a time. Instead, for any $q \in \mathbf{N}$ large enough, the martingales $d^{i,j}_{k';q}$ for all $k' \geq i$ will act together to "succeed as a group" on all sets to which A^i reduces via F^i_j.

The martingale $d^{i,j}_{k;q}$ will be split up into infinitely many martingales

$$d^{i,j}_{k;q} = \sum_{\ell=1}^{\infty} d^{i,j}_{k;q;\ell},$$

where each martingale $d^{i,j}_{k;q;\ell}$ bets a finite number of times. Fix i and j. For any $m \in \mathbf{N}$, let y_m be least such that $v < y_m$ for all queries $\langle u, v \rangle$ made by F^i_j on inputs $\langle j, x \rangle$ for all $x < m$. For any $C \subseteq \mathbf{N}$, let $\mathbf{E}^C(m)$ be the event that

$F^i_j(C)[j, < m] = A^i[j, < m]$, i.e., that $F^i_j(C)$ and A^i agree on the first m elements of the jth column. For all $w \in \{0,1\}^*$, we define

$$d^{i,j}_{k;q;\ell}(w) = 2^{|w|-\ell} \cdot \Pr_C \left[\xi^H(k, y_{q\ell}) \cup (\{k\} \times w) \sqsubset C \mid \xi^H(k, y_{q\ell}) \sqsubset C \ \ \& \ \ \mathbf{E}^C(q\ell) \right]$$

if $\Pr_C \left[\xi^H(k, y_{q\ell}) \sqsubset C \ \ \& \ \ \mathbf{E}^C(q\ell) \right] > 0$. Otherwise, for all w define $d^{i,j}_{k;q;\ell}(w) = 2^{-\ell}$.

We now define H_k. For any y, we assume that $H_k[< y]$ has already been defined, and we set $w = H_k[< y]$. Let

$$H_k(y) = \begin{cases} \alpha_k(\langle k, y \rangle) & \text{if } \alpha_k(\langle k, y \rangle) \text{ is defined,} \\ 0 & \text{if } \alpha_k(\langle k, y \rangle) \text{ is undefined and } d'_k(w0) \leq d'_k(w1), \\ 1 & \text{if } \alpha_k(\langle k, y \rangle) \text{ is undefined and } d'_k(w0) > d'_k(w1). \end{cases}$$

Remark. Actually, we cannot do this exactly as stated. A recursive martingale such as d'_k cannot in general be computed exactly, but is only approximated. What we are really comparing are not $d'_k(w0)$ and $d'_k(w1)$, but rather their yth approximations, which *are* computable. Since these approximations are guaranteed to be within 2^{-y} of the actual values, and our sole aim is to make d'_k fail on H_k, it suffices for our purposes to consider only the approximations when doing the comparisons above. The same trick is used in [8].

H_k is evidently recursive (given the last remark), and for cofinitely many y, $H_k(y)$ is chosen so that $d'_k(H_k[< (y+1)]) \leq d'_k(H_k[< y]) + 2^{-y}$, the 2^{-y} owing to the error in the approximation of d'_k. Thus d'_k fails on H_k, from which we obtain

Fact 2 *The martingales $\tilde{d}_k$ and $d^{i,j}_{k;q}$ for all and $i \leq k$, j, and q all fail on H_k.*

Thus Conditions 1 and 2 are satisfied. Each H_k also preserves the diagonalization commitments made by the $\alpha_{k'}$ for all $k' \leq k$, so the following is easily checked:

Fact 3 $\alpha_0 \sqsubseteq \alpha_1 \sqsubseteq \alpha_2 \sqsubseteq \cdots \sqsubset H$.

To verify Condition 3, we show that $A^i \neq F^i_j(H)$ for all i and j. Suppose $A^i = F^i_j(H)$ for some i and j. Let $k_0 = \langle i, j \rangle$, and let $\sigma = H_{<k_0} \cup \alpha_{k_0 - 1}$. By the definition of α_{k_0}, it must be the case that $A^i = F^i_j(C)$ for all $C \sqsupset \sigma$, otherwise F^i_j would have been diagonalized against by α_{k_0} and would thus fail to reduce A^i to H. Let q_0 be smallest such that $q_0 > i$ and $\sigma(\langle q', y \rangle)$ is undefined for all y and $q' \geq q_0$. We will show that $d^{i,j}_{n;q_0}$ succeeds on H_n for some $n < q_0$, contradicting Fact 2 above.

For any $C \subseteq \mathbf{N}$ and $m \in \mathbf{N}$, we let y_m and $\mathbf{E}^C(m)$ be as before. For any ℓ and $y \geq y_{q_0\ell}$ we have

$$\Pr_C \left[\mathbf{E}^C(q_0\ell) \mid \xi^H(q_0, y) \sqsubset C \right] = 1$$

by the definition of q_0 and $y_{q_0\ell}$, and thus

$$\begin{aligned}
&\frac{\Pr_C\left[\mathbf{E}^C(q_0\ell) \mid \xi^H(q_0,y) \sqsubset C\right]}{\Pr_C\left[\mathbf{E}^C(q_0\ell) \mid \xi^H(i,y) \sqsubset C\right]}\\
&= \frac{1}{\Pr_C\left[\mathbf{E}^C(q_0\ell) \mid \xi^H(i,y) \sqsubset C\right]}\\
&= \frac{1}{\Pr_C\left[\mathbf{E}^C(q_0\ell)\right]}\\
&\geq 2^{q_0\ell},
\end{aligned}$$

the last inequality following from Fact 1. From the definition of $d^{i,j}_{k;q_0;\ell}$, the following inequation can be shown for any ℓ and $y \geq y_{q_0\ell}$ (details are omitted)

$$\prod_{k=i}^{q_0-1} d^{i,j}_{k;q_0;\ell}(H_k[< y]) \geq 2^{-q_0\ell} \cdot \frac{\Pr_C\left[\mathbf{E}^C(q_0\ell) \mid \xi^H(q_0,y) \sqsubset C\right]}{\Pr_C\left[\mathbf{E}^C(q_0\ell) \mid \xi^H(i,y) \sqsubset C\right]}$$

Therefore,

$$\prod_{k=i}^{q_0-1} d^{i,j}_{k;q_0;\ell}(H_k[< y]) \geq 1$$

for all $y \geq y_{q_0\ell}$, which implies that $d^{i,j}_{k;q_0;\ell}(H_k[< y]) \geq 1$ for at least one k between i and $q_0 - 1$. Since q_0 is fixed and ℓ was chosen arbitrarily, by the Pigeon-Hole Principle there must be some n_0 with $i \leq n_0 < q_0$ such that for infinitely many ℓ, $d^{i,j}_{n_0;q_0;\ell}(H_{n_0}[< y]) \geq 1$ for all $y \geq y_{q_0\ell}$. This in turn implies that the martingale $d^{i,j}_{n_0;q_0}$ succeeds on H_{n_0}, contradicting Fact 2.

Thus $A^i \neq F^i_j(H)$ for all i and j, and Condition 3 is satisfied.

□ **Theorem 12**

Corollary 14. *There is a sequence that is strongly deep but not strongly useful.*

Proof. This follows immediately from Theorem 12 and the fact [6] that every weakly useful sequence is strongly deep. □

It is easy to verify that weak and strong usefulness are both invariant under tt-equivalence. Thus, Theorem 12 shows that there are weakly useful tt-degrees that are not strongly useful. Our results do not say anything regarding the *Turing* degrees of weakly useful sets, however. In particular, we leave open the question of whether there is a weakly useful Turing degree that is not strongly useful (i.e., whether there is a weakly useful set not Turing equivalent to any strongly useful set). Some facts are known about these degrees. Jockusch [4] neatly characterized the *strongly* useful Turing degrees (under a different name), for example, as being either high or containing complete extensions of first-order Peano arithmetic. This includes some low degrees, but no non-high r.e. degrees. Recently, Stephan [14] has partially strengthened these results, showing that no non-high r.e. Turing degree can be *weakly* useful, either. Therefore, among the r.e. degrees, the strongly useful, weakly useful, and high degrees all coincide.

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References

1. Miklos Ajtai and Ronald Fagin, Reachability is harder for directed than for undirected graphs, *Journal of Symbolic Logic* **55** (1990), pp. 113–150.
2. Y. M. Barzdin′, Complexity of programs to determine whether natural numbers not greater than n belong to a recursively enumerable set, *Soviet Mathematics Doklady* **9** (1968), pp. 1251–1254.
3. C. H. Bennett, Logical depth and physical complexity, In R. Herken, editor, *The Universal Turing Machine: A Half-Century Survey*, pp. 227–257. Oxford University Press, 1988.
4. Jr. C. G. Jockusch, Degrees in which the recursive sets are uniformly recursive, *Canadian Journal of Mathematics* **24** (1972), pp. 1092–1099.
5. Jr. C. J. Jockusch and R. I. Soare, Degrees of members of Π^0_1 classes, *Pacific Journal of Mathematics* **40** (1972), pp. 605–616.
6. David W. Juedes, James I. Lathrop, and Jack H. Lutz, Computational depth and reducibility, *Theoretical Computer Science* **132** (1994), pp. 37–70, also appeared in *Proceedings of the Twentieth International Colloquium on Automata, Languages, and Programming*, 1993, pp. 278–288.
7. M. Li and P. M. B. Vitányi, *An Introduction to Kolmogorov Complexity and its Applications*, Springer, 1993.
8. J. H. Lutz, Weakly hard problems, *SIAM Journal on Computing* to appear, also appeared in *Proceedings of the Ninth Structure in Complexity Theory Conference*, 1993, pp. 146–161.
9. J. H. Lutz, Almost everywhere high nonuniform complexity, *Journal of Computer and System Sciences* **44** (1992), pp. 220–258.
10. P. Martin-Löf, On the definition of random sequences, *Information and Control* **9** (1966), pp. 602–619.
11. E. L. Post, Recursively enumerable sets of positive integers and their decision problems, *Bulletin of the American Mathematical Society* **50** (1944), pp. 284–316.
12. H. Rogers, Jr, *Theory of Recursive Functions and Effective Computability*, McGraw - Hill, 1967.
13. G. E. Sacks, *Degrees of Unsolvability*, Princeton University Press, 1966.
14. F. Stephan, 1994, private communication.

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