Feasible Reductions to Kolmogorov–Loveland Stochastic Sequences [1]

Jack H. Lutz
Department of Computer Science
Iowa State University
Ames, IA 50011
U.S.A.
lutz@cs.iastate.edu

David L. Schweizer
Department of Mathematics
College of the Holy Cross
Worcester, MA 01610
U.S.A.
davids@math.holycross.edu

Abstract

For every binary sequence A, there is an infinite binary sequence S such that $A \leq^{\mathrm{P}}_{\mathrm{tt}} S$ and S is stochastic in the sense of Kolmogorov and Loveland.

1 Introduction

In the mid-1960's, Martin-Löf [23] used the general theory of algorithms to formulate the first successful definition of the randomness of individual binary sequences. Subsequent definitions, using a variety of conceptual approaches, were introduced by Levin [17], Schnorr [24, 25], Chaitin [6, 7, 8], Solovay [28], and Shen′ [26]. Each of these definitions was shown to be equivalent to Martin-Löf's, in the sense that a binary sequence R is algorithmically random according to the given definition if and only if R is algorithmically random according to Martin-Löf's definition.

In the present note, all "sequences" are infinite binary sequences, and the term "random" means "algorithmically random in the sense of Martin-Löf". A precise definition of algorithmic randomness appears in section 2.

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One of the most useful and intuitive properties of random sequences is their stochasticity. This is the fact that if a subsequence A of a random sequence R is chosen according to an "admissible selection rule", then the limiting frequency of 1's in the subsequence A is exactly $\frac{1}{2}$. The broadest class of admissible selection rules that has been studied in this context is the class of *Kolmogorov–Loveland selection rules* [13, 14, 19, 20]. These algorithmic rules (which are described in section 2) are more general than earlier selection rules proposed by von Mises [31], Wald [32], and Church [9] in two respects. First, given a sequence S, a Kolmogorov–Loveland selection rule may choose bits from S in whatever order arises from the rule's interaction with S; this order need not agree with the order of appearance of these bits in S. Second, a Kolmogorov–Loveland selection rule is a partial recursive rule that may succeed in choosing a sequence of distinct bits from one sequence, yet fail to choose such a sequence from another.

It is easy to see that every random sequence R is *Kolmogorov–Loveland stochastic*. This means that, for every sequence A of distinct bits of R that is chosen according to a Kolmogorov–Loveland selection rule, the limiting frequency of 1's in A is $\frac{1}{2}$. In the late 1980's, Shen′ [27] proved that the converse does not hold, thereby solving a problem that had been open for some twenty years. (See [15, 29, 18] for more detailed histories of this problem and the role of stochasticity in the foundations of probability theory.) Thus, the random sequences form a proper subset of the set of all Kolmogorov–Loveland stochastic sequences.

This note refines the method of Shen′ [27] in order to establish a stronger, more quantitative separation between randomness and Komogorov–Loveland stochasticity.

Kučera [16] and Gács [10] have proven that for every sequence A there is a random sequence R such that A is Turing reducible to R. However, it is well known that this does not hold for truth-table reducibility (Turing reducibility with computable running time). In fact, Juedes, Lathrop, and Lutz [12] have noted that, in the sense of Baire category, *almost every* sequence A has the property that A is not reducible to *any* random sequence in *any* computable running time.

In contrast with this fact, the main theorem of the present note (Theorem 3.5) states that, for *every* sequence A, there is a Kolmogorov–Loveland stochastic sequence S such that A is *feasibly* reducible to S. In fact, A can be

reduced to S by a polynomial-time truth-table reduction. The proof of this result uses a relativization of a method of van Lambalgen [30] and Shen′ [27], together with a simple encoding of the sequence A into a "nearly uniform" probability measure on the set of all sequences.

It follows immediately from the main theorem that there are sequences S that are Kolmogorov–Loveland stochastic, but also strongly deep in the sense of Bennett [1]. Such sequences S are computationally "very far from random" [1, 12].

The main theorem also implies that the class RAND of all random oracles cannot be replaced by the class KL-STOCH of all Kolmogorov–Loveland stochastic oracles in some known characterizations of complexity classes. As just one example, a "folklore" result states that $\mathrm{P}(\mathrm{RAND}) \cap \mathrm{REC} = \mathrm{BPP}$, that is, that a recursive language is $\leq^{\mathrm{P}}_{\mathrm{T}}$-reducible to some random language if and only if it is probabilistically decidable with bounded error in polynomial time [1, 3]. In contrast, the main theorem immediately implies that P(KL-STOCH) contains *every* language. See [3, 22, 2, 4] for other known characterizations using random oracles that, by the main theorem, cannot be extended to Kolmogorov–Loveland stochastic oracles.

2 Notation and Preliminaries

We write $\{0,1\}^*$ for the set of all (finite, binary) *strings*, and we write $|x|$ for the length of a string x. The empty string, λ, is the unique string of length 0. The *standard enumeration* of $\{0,1\}^*$ is the sequence $s_0 = \lambda, s_1 = 0, s_2 = 1, s_3 = 00, \ldots$, ordered first by length and then lexicographically.

The *complement* of a language A is $A^c = \{0,1\}^* - A$, and the symmetric difference of languages A and B is $A \triangle B = (A - B) \cup (B - A)$.

The *Boolean value* of a condition ϕ is $[\![\phi]\!] = \mathbf{if}\ \phi\ \mathbf{then}\ 1\ \mathbf{else}\ 0$.

We work in the *Cantor space* $\mathbf{C}$, consisting of all languages $A \subseteq \{0,1\}^*$. We identify each language A with its *characteristic sequence*, which is the (infinite, binary) sequence A whose n^{th} bit is $[\![s_n \in A]\!]$ for each $n \in \mathbb{N}$. (The leftmost bit of A is the 0^{th} bit.) Relying on this identification, we also consider $\mathbf{C}$ to be the set of all sequences.

A string w is a *prefix* of a sequence A, and we write $w \sqsubseteq A$, if there is a sequence B such that $A = wB$. For each string $w \in \{0,1\}^*$, the *cylinder generated by* w is the set

$$\mathbf{C}_w = \left\{ A \in \mathbf{C} \;\middle|\; w \sqsubseteq A \right\}.$$

Note that $\mathbf{C}_\lambda = \mathbf{C}$.

Let D be a discrete domain such as $\mathbb{N}$, $\{0,1\}^*$, or $\mathbb{N} \times \{0,1\}^*$. A function $f : D \to \mathbb{R}$ is *computable* if there is a total recursive function $\hat{f} : \mathbb{N} \times D \to \mathbb{Q}$ such that, for all $r \in \mathbb{N}$ and $x \in D$, $|\hat{f}(r,x) - f(x)| \leq 2^{-r}$. A function $f : D \to \mathbb{R}$ is is *lower semicomputable* if there is a total recursive function $\hat{f} : \mathbb{N} \times D \to \mathbb{Q}$ such that (i) for all $r \in \mathbb{N}$ and $x \in D$, $\hat{f}(r,x) \leq \hat{f}(r+1,x)$, and (ii) for all $x \in D$, $\lim_{r\to\infty} \hat{f}(r,x) = f(x)$. A sequence $(\beta_0, \beta_1, \ldots)$ of real numbers *converges computably* to a limit $\beta \in \mathbb{R}$ if there is a total recursive function $m : \mathbb{N} \to \mathbb{N}$, called a *modulus of convergence,* such that, for all $r \in \mathbb{N}$ and $i \geq m(r)$, $|\beta_i - \beta| \leq 2^{-r}$. Similarly, a series $\sum_{n=0}^{\infty} \alpha_n$ of nonnegative reals α_n is *computably convergent* if there is a total recursive function (modulus of convergence) $m : \mathbb{N} \to \mathbb{N}$ such that, for all $r \in \mathbb{N}$, $\sum_{n=m(r)}^{\infty} \alpha_n \leq 2^{-r}$.

A *bias sequence* is a sequence $\vec{\beta} = (\beta_0, \beta_1, \ldots)$ of real numbers (biases) $\beta_i \in [0,1]$. A bias sequence $\vec{\beta}$ determines the *coin-toss probability measure* $\mu^{\vec{\beta}}$ on Cantor space, which corresponds to a random experiment in which a language $A \in \mathbf{C}$ is chosen probabilistically as follows. For each string s_i, we toss a special coin whose probability is β_i of coming up heads, in which case $s_i \in A$, and $1 - \beta_i$ of coming up tails, in which case $s_i \notin A$. The coin tosses are independent of one another. In the special case where $\vec{\beta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\right)$, $\mu^{\vec{\beta}}$ is the *uniform probability measure* on $\mathbf{C}$.

As noted in the introduction, there are several equivalent definitions of algorithmic randomness. The definition in terms of martingales, introduced by Schnorr [24], is most convenient for our purposes here. Given a bias sequence $\vec{\beta}$, a *$\vec{\beta}$-martingale* is a function $d : \{0,1\}^* \to [0,\infty)$ such that for all $w \in \{0,1\}^*$,

$$d(w) = (1 - \beta_{|w|})d(w0) + \beta_{|w|}d(w1).$$

(The reader is referred to [30, 5] for discussion of this definition and its motivation.) The *success set* of a $\vec{\beta}$-martingale d is

$$S^\infty[d] = \left\{ A \in \mathbf{C} \;\middle|\; (\forall k)(\exists w \sqsubseteq A)\; d(w) \geq k \right\}.$$

The *unitary success set* of a $\vec{\beta}$-martingale d is

$$S^1[d] = \bigcup_{d(w)\geq 1} \mathbf{C}_w.$$

A sequence $R \in \mathbf{C}$ is *(algorithmically) $\vec{\beta}$-random,* and we write $R \in \mathrm{RAND}_{\vec{\beta}}$, if there is no lower semicomputable $\vec{\beta}$-martingale d such that $R \in S^\infty[d]$. A sequence $R \in \mathbf{C}$ is rec-$\vec{\beta}$*-random,* and we write $R \in \mathrm{RAND}_{\vec{\beta}}(\mathrm{rec})$, if there is no computable $\vec{\beta}$-martingale d such that $R \in S^\infty[d]$. It is immediate from the definitions that $\mathrm{RAND}_{\vec{\beta}} \subseteq \mathrm{RAND}_{\vec{\beta}}(\mathrm{rec})$. When the probability measure is uniform, that is, $\vec{\beta} = \left(\frac{1}{2}, \frac{1}{2}, \ldots\right)$, we omit $\vec{\beta}$ from the notation. Thus RAND is the set of all *random* sequences, and RAND(rec) is the set of all rec-*random* sequences.

It is a straightforward matter to relativize the computability or lower semicomptability of a martingale d to an arbitrary oracle $A \in \mathbf{C}$, and thereby to define the class $\mathrm{RAND}^A_{\vec{\beta}}$, consisting of all sequences that are $\vec{\beta}$*-random relative to* A, and the class $\mathrm{RAND}^A_{\vec{\beta}}(\mathrm{rec})$, consisting of all sequences that are rec-$\vec{\beta}$*-random relative to* A.

The following property of rec-$\vec{\beta}$-random sequences (relative to an oracle A) is an easy extension of a special case of the resource-bounded Borel-Cantelli lemma of [21].

Lemma 2.1. Let $A \in \mathbf{C}$ and let $\vec{\beta}$ be a bias sequence that is computable relative to A. Let $d_0, d_1, d_2, \ldots$ be a sequence of $\vec{\beta}$-martingales with the following two properties:

(i) The function $(n, w) \mapsto d_n(w)$ is computable relative to A.

(ii) The series $\sum_{n=0}^\infty d_n(\lambda)$ is computably convergent relative to A.

If $R \in \mathrm{RAND}^A_{\vec{\beta}}(\mathrm{rec})$, then there are only finitely many n for which $R \in S^1[d_n]$.

Proof. Assume the hypothesis, and let $J = \left\{ n \;\middle|\; R \in S^1[d_n] \right\}$. Let $m : \mathbb{N} \to \mathbb{N}$ be a modulus for the convergence of $\sum_{n=0}^\infty d_n(\lambda)$ that is total recursive

relative to A, and define the function $d : \{0,1\}^* \to [0,\infty)$ by

$$d(w) = \sum_{r=0}^{\infty} 2^r \sum_{n=m(2r)}^{\infty} d_n(w).$$

It is easily checked that d is a $\vec{\beta}$-martingale that is computable relative to A. Since $R \in \mathrm{RAND}^A_{\vec{\beta}}(\mathrm{rec})$, it follows that there is a constant $c \in \mathbb{N}$ such that, for all $w \sqsubseteq R$, $d(w) < 2^c$.

Now let $n_0 \in J$. Fix a prefix $w \sqsubseteq R$ such that $d_{n_0}(w) \geq 1$. Then we have

$$2^c d_{n_0}(w) \geq 2^c > d(w) \geq 2^c \sum_{n=m(2c)}^{\infty} d_n(w),$$

so $n_0 < m(2c)$. Thus J is finite. □

The notion of Kolmogorov–Loveland stochasticity was defined in [13, 14, 19, 20]; detailed discussions may be found in [29, 15, 18]. A sequence is *Kolmogorov–Loveland stochastic* if any subsequence chosen by a Kolmogorov–Loveland selection rule possesses frequency stability, that is, if the proportion of 1's in initial segments tends toward a limit of $\frac{1}{2}$. A *Kolmogorov–Loveland selection rule* is a pair of partial recursive functions that, operating on the history of what has been observed, choose the index of the next bit of the sequence to examine and determine (in advance of examination) whether or not that bit will be included in the subsequence. The standard intuition is described elegantly in [27]:

> Let us imagine that the members of a sequence are written on cards which lie on an (infinitely long) table (we do not see what is on a card unless we turn it). The [selection rule] is an algorithm that says which card must be turned next and whether it must be turned only for information or [is to be] selected into the subsequence.

We also make use of the following large deviation result.

Lemma 2.2 (Chernoff bound [11]). Let $p \in [0,1]$, let $X_1, \ldots, X_n$ be independent 0/1-valued random variables such that each $P[X_i = 1] = p$, and let $S = X_1 + \cdots + X_n$. Then:

1. For all $0 \leq \epsilon \leq 1, P[S \geq (1+\epsilon)np] \leq e^{-\frac{\epsilon^2 np}{3}}$.

2. For all $0 \leq \epsilon < 1, P[S \leq (1-\epsilon)np] \leq e^{-\frac{\epsilon^2 np}{2}}$.

3 Result

In this section we prove that every sequence is feasibly reducible to some Kolmogorov–Loveland stochastic sequence. Our proof makes essential use of the following lemma, which is a straightforward relativization of Lemma 2 of [27].

Lemma 3.1 (Shen′ [27]). Let $A \in \mathbf{C}$, and let $\vec{\beta}$ be a sequence of biases such that

(i) $\vec{\beta}$ is computable relative to A, and

(ii) $\vec{\beta}$ converges computably to $\frac{1}{2}$ relative to A.

Then $\mathrm{RAND}^A_{\vec{\beta}} \subseteq \text{KL-STOCH}^A$.

Given a sequence $A \in \mathbf{C}$, the following construction defines a sequence of biases $\vec{\beta}^A$, a function $F^A : \mathbf{C} \to \mathbf{C}$, and some auxiliary notation.

Construction 3.2. We use the functions

$$l, q, r : \mathbb{N} \to \mathbb{N},$$

where

$$l(n) = |s_n| = \lfloor \log(n+1) \rfloor,$$

$$q(n) = 12\Big(l(n)+3\Big)^3,$$

and

$$r(n) = \sum_{m=0}^{n-1} q(m).$$

We also use the function

$$\epsilon : \mathbb{N} \to [0,1]$$

where

$$\epsilon(n) = \frac{1}{l(n)+3}.$$

For $A \in \mathbf{C}$ and $i \in \mathbb{N}$, define the bias $\beta_i^A \in [0,1]$ by

$$\beta_i^A = \frac{1}{2}\Big(1 + \ 3\epsilon(n)[\![s_n \in A]\!]\Big),$$

where $r(n) \leq i < r(n) + q(n)$, and let

$$\vec{\beta}^A = (\beta_0^A, \beta_1^A, \ldots).$$

(Note that the bit $[\![s_n \in A]\!]$ has been encoded into each of $q(n)$ different positions in the bias sequence $\vec{\beta}^A$.) For each $A \in \mathbf{C}$ and $n \in \mathbb{N}$, define the random variable

$$\rho_n^A : \mathbf{C} \to [0,1]$$

by

$$\rho_n^A(S) = \frac{1}{q(n)}\left|\left\{ i \in \mathbb{N} \;\middle|\; r(n) \leq i < r(n) + q(n) \ \text{ and } \ s_i \in S \right\}\right|,$$

where the argument S is chosen probabilistically according to the bias sequence $\vec{\beta}^A$. Finally, for each $A \in \mathbf{C}$, define the function

$$F^A : \mathbf{C} \to \mathbf{C}$$

by

$$F^A(S) = \left\{ s_n \;\middle|\; \rho_n^A(S) \geq \frac{1}{2}\Big(1 + \epsilon(n)\Big) \right\}.$$

Although the probability distribution of the random variable ρ_n^A depends on the sequence A, the actual value $\rho_n^A(S)$ does not depend on A. Moreover, we have the following.

Observation 3.3. For all $A, S \in \mathbf{C}$,

$$F^A(S) \leq_{\mathrm{tt}}^{\mathrm{P}} S.$$

Proof. The bit $[\![s_n \in F^A(S)]\!]$ is easily computed from the $q(n)$ bits

$$[\![s_{r(n)} \in S]\!], [\![s_{r(n)+1} \in S]\!], \ldots, [\![s_{r(n)+q(n)-1} \in S]\!],$$

and $q(n)$ is polynomial in $|s_n|$. □

Intuitively, if S is chosen according to the bias sequence $\vec{\beta}^A$, then we expect $\rho_n^A(S)$ to be approximately $\frac{1}{2}(1+3\epsilon(n))$ if $n \in A$, and approximately $\frac{1}{2}$ if $n \notin A$. Since $\frac{1}{2} < \frac{1}{2}(1+\epsilon(n)) < \frac{1}{2}(1+3\epsilon(n))$, we thus expect that $[\![s_n \in F^A(S)]\!]$ will usually agree with $[\![s_n \in A]\!]$. The following lemma formalizes this intuition.

Lemma 3.4. For all $A \in \mathbf{C}$ and $S \in \mathrm{RAND}^A_{\vec{\beta}^A}(\mathrm{rec})$,

$$\Big| A \triangle F^A(S) \Big| < \infty.$$

Proof. Let $A \in \mathbf{C}$ and $S \in \mathrm{RAND}^A_{\vec{\beta}^A}(\mathrm{rec})$. For convenience, we write $\vec{\beta} = \vec{\beta}^A$, $\rho_n = \rho_n^A$, and $F = F^A$. For each $n \in \mathbb{N}$, define the event

$$\mathcal{E}_n = \Big\{ B \in \mathbf{C} \;\Big|\; n \in A \triangle F(B) \Big\}$$

and the function

$$d_n : \{0,1\}^* \to [0,1],$$

by

$$d_n(w) = P(\mathcal{E}_n \mid \mathbf{C}_w),$$

where the conditional probability refers (as do all subsequent probabilities in this proof) to the coin-toss probability measure $\mu^{\vec{\beta}}$. It is routine to check that each d_n is a $\vec{\beta}$-martingale, and that the function $(n,w) \mapsto d_n(w)$ is computable relative to A.

By the Chernoff bound (Lemma 2.2), for each $s_n \in A$, we have

$$\begin{aligned}
P[s_n \notin F(S)] &= P\Big[\rho_n < \frac{1}{2}\Big(1+\epsilon(n)\Big)\Big] \\
&\leq P\Big[\rho_n < \frac{1}{2}\Big(1+3\epsilon(n)\Big)\Big(1-\epsilon(n)\Big)\Big] \\
&\leq e^{-\frac{\epsilon(n)^2 q(n)(1+3\epsilon(n))}{4}} \\
&< e^{-\frac{\epsilon(n)^2 q(n)}{4}} \\
&\leq (n+1)^{-3}.
\end{aligned}$$

Similarly, for each $s_n \in A^c$, we have

$$\begin{aligned} P[s_n \in F(S)] &= P\Big[\rho_n \geq \frac{1}{2}\Big(1+\epsilon(n)\Big)\Big] \\ &\leq e^{-\frac{\epsilon(n)^2 q(n)}{6}} \\ &\leq (n+1)^{-2}. \end{aligned}$$

Hence, for each $n \in \mathbb{N}$,

$$d_n(\lambda) = P(\mathcal{E}_n) \leq (n+1)^{-3} + (n+1)^{-2}.$$

Since the series

$$\sum_{n=0}^{\infty} \Big[(n+1)^{-3} + (n+1)^{-2}\Big]$$

is computably convergent, it follows by Lemma 2.1 that the set

$$J = \Big\{ s_n \;\Big|\; S \in S^1[d_n] \Big\}.$$

is finite. Since the definition of d_n implies that $A \triangle F(S) \subseteq J$, this completes the proof of the lemma. □

Our main result is now easily established.

Theorem 3.5 (Main Theorem). For each $A \in \mathbf{C}$, there is a sequence $S \in$ KL-STOCH such that $A \leq_{tt}^{P} S$.

Proof. Let $A \in \mathbf{C}$, and let $S \in \mathrm{RAND}^A_{\vec{\beta}^A}$. By Lemma 3.1, $S \in \text{KL-STOCH}^A \subseteq$ KL-STOCH. Also, since $\mathrm{RAND}^A_{\vec{\beta}^A} \subseteq \mathrm{RAND}^A_{\vec{\beta}^A}(\mathrm{rec})$, Lemma 3.4 tells us that $|A \triangle F^A(S)| < \infty$, whence $A \leq^{\mathrm{P}}_{\mathrm{tt}} F^A(S)$. It follows by Observation 3.3 that $A \leq^{\mathrm{P}}_{\mathrm{tt}} S$. □

As noted in the introduction, Theorem 3.5 exhibits a strong, quantitative separation of RAND from KL-STOCH, since Juedes, Lathrop, and Lutz [12] have shown that only a meager (that is, negligibly small in the sense of Baire category) set of sequences have the property of being reducible to some random sequence in some computable running time.

Bennett [1] has introduced the notion of *computational depth,* which measures the "value" of information in terms of the amount of "computational work" that has been "added to its organization". In the case of infinite binary sequences, Bennett has defined both *strong depth* and *weak depth,* and shown that, in a technical, computational sense, strongly deep sequences are "very far from random". (See [1, 18, 12] for definitions and discussion of computational depth.)

The proof of Shen′ [27] exhibits a sequence that is Kolmgorov–Loveland stochastic and not random. That sequence, however, is random relative to a computable probability measure and so is not even weakly deep in the sense of Bennett. Nevertheless, we now note that a Kolmogorov–Loveland stochastic sequence may be strongly deep.

Corollary 3.6. There exist Kolmogorov–Loveland stochastic sequences that are strongly deep.

Proof. Let K be the diagonal halting problem. By Theorem 3.5, let S be a Kolmogorov–Loveland stochastic sequence such that $K \leq_{\mathrm{tt}}^{\mathrm{P}} S$. Bennett [1] has shown that K is strongly deep and, by his deterministic slow growth law, that strongly deep sequences are only $\leq_{\mathrm{tt}}$-reducible to sequences that are themsleves strongly deep. (Proofs of these results also appear in [12].) Hence, S is strongly deep. □

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