Effective Fractal Dimensions

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1 Introduction

The two most important notions of fractal dimension are *Hausdorff dimension*, developed by Hausdorff [17], and *packing dimension*, developed by Tricot [63] and Sullivan [60]. Both dimensions have the mathematical advantage of being defined from measures, and both have yielded extensive applications in fractal geometry and dynamical systems. In 2000, Lutz [32] proved a simple characterization of Hausdorff dimension in terms of *gales*, which are betting strategies that generalize martingales. Imposing various computability and complexity constraints on these gales produces a spectrum of effective versions of Hausdorff dimension, including constructive, computable, polynomial-space, polynomial- time, and finite-state dimensions. Work by several investigators has already used these effective dimensions to shed light on a variety of topics in theoretical computer science, including algorithmic information theory, computational complexity, prediction, and data compression. Constructive dimension has also been discretized, assigning a dimension $\dim(x)$ to each string $x \in \{0,1\}^*$ in a way that arises naturally from Hausdorff and constructive dimensions and gives the unexpected characterization $K(x) = |x|\dim(x) \pm O(1)$ of Kolmogorov complexity. More recently, Athreya, Hitchcock, Lutz, and Mayordomo [3] proved that packing dimension – previously thought to be much more complex than Hausdorff dimension [40] – admits a gale characterization that is an exact dual of that of Hausdorff dimension. We survey these developments and their implications for the theory of computing.

Portions of this work have been surveyed earlier by Mayordomo [41] and Terwijn [62]. An online bibliography on effective fractal dimensions is maintained by Hitchcock [18].

2 Gales and Fractal Dimensions

Effective fractal dimensions were formulated by the following two steps.

1. Characterize classical fractal dimensions - which were originally defined in other terms - in terms of certain betting strategies, called *gales*.

2. Impose computability or complexity constraints on these gales.

In this section we review the first of these steps.

Notation. The *Cantor space* **C** is the set of all infinite binary sequences. The n-bit *prefix* of a sequence $S \in \mathbf{C}$ is the binary string $S[0..n-1]$.

Definition. Let $s \in [0, \infty)$.

1. An *s-supergale* is a function $d : \{0, 1\}^* \longrightarrow [0, \infty)$ that satisfies the condition

$$d(w) \geq 2^{-s}[d(w0) + d(w1)] \tag{2.1}$$

   for all $w \in \{0, 1\}^*$.

2. An *s-gale* is an s-supergale that satisfies (2.1) with equality for all $w \in \{0, 1\}^*$.

3. A *supermartingale* is a 1-supergale.

4. A *martingale* is a 1-gale.

Martingales, introduced by Lévy [30] and Ville [64] have been used extensively by Schnorr [48, 49, 50] and others in the investigation of randomness and by Lutz [34, 36] and others in the development of resource-bounded measure.

Intuitively, we regard a supergale d as a strategy for betting on the successive bits of a sequence $S \in \mathbf{C}$. More specifically $d(w)$ is the amount of capital that d has after betting on the prefix w of S. If $s = 1$, then the right-hand side of (2.1) is the conditional expectation of $d(wb)$ given that w has occurred (when b is a uniformly distributed binary random variable). Thus a martingale models a gambler's capital when the payoffs are fair. (The expected capital after the bet is the actual capital before the bet.) In the case of an s-gale, if $s < 1$, the payoffs are less than fair; if $s > 1$, the payoffs are more than fair.

We now define two criteria for the success of a gale or supergale.

Definition. Let d be an s-supergale, where $s \in [0, \infty)$.

1. We say that d *succeeds* on a sequence $S \in \mathbf{C}$ if

$$\limsup_{n \to \infty} d(S[0..n-1]) = \infty. \tag{2.2}$$

   The *success set* of d is $S^{\infty}[d] = \{S \in \mathbf{C} | d \text{ succeeds on } S\}$.

2. We say that d *succeeds strongly* on a sequence $S \in \mathbf{C}$ if

$$\liminf_{n \to \infty} d(S[0..n-1]) = \infty. \tag{2.3}$$

   The *strong success set* of d is $S^{\infty}_{\mathrm{str}}[d] = \{S \in \mathbf{C} | d \text{ succeeds strongly on } S\}$.

We have written conditions (2.2) and (2.3) in a fashion that emphasizes their duality. Condition (2.2) says simply that the set of values $d(S[0..n-1])$ is unbounded, while condition (2.3) says that $d(S[0..n-1]) \to \infty$ as $n \to \infty$.

Notation. Let $X \subseteq \mathbf{C}$.

1. $\mathcal{G}(X)$ is the set of all $s \in [0, \infty)$ for which there exists an s-gale d such that $X \subseteq S^{\infty}[d]$.

2. $\mathcal{G}^{\mathrm{str}}(X)$ is the set of all $s \in [0, \infty)$ for which there exists an s-gale d such that $X \subseteq S^{\infty}_{\mathrm{str}}[d]$.

3. $\widehat{\mathcal{G}}(X)$ is the set of all $s \in [0, \infty)$ for which there exists an s-supergale d such that $X \subseteq S^\infty[d]$.

4. $\widehat{\mathcal{G}}^{\mathrm{str}}(X)$ is the set of all $s \in [0, \infty)$ for which there exists an s-supergale d such that $X \subseteq S^\infty_{\mathrm{str}}[d]$.

Note that $s' \geq s \in \mathcal{G}(X)$ implies that $s' \in \mathcal{G}(X)$, and similarly for the classes $\mathcal{G}^{\mathrm{str}}(X)$, $\widehat{\mathcal{G}}(X)$, and $\widehat{\mathcal{G}}^{\mathrm{str}}(X)$. The following fact is also clear.

Observation 2.1. *For all $X \subseteq \mathbf{C}$, $\mathcal{G}(X) = \widehat{\mathcal{G}}(X)$ and $\mathcal{G}^{\mathrm{str}}(X) = \widehat{\mathcal{G}}^{\mathrm{str}}(X)$.*

Each set $X \subseteq \mathbf{C}$ has a *Hausdorff dimension* $\dim_{\mathrm{H}}(X)$, defined by Hausdorff [17], and a *packing dimension* $\dim_{\mathrm{P}}(X)$, defined independently by Tricot [63] and Sullivan [60]. These definitions appear in many standard texts, e.g., [14, 13], but they are not used in the present survey, so we do not reproduce them. For our purpose here, it suffices to use the following characterizations as the definitions of the Hausdorff and packing dimensions in $\mathbf{C}$.

Theorem 2.2 (Gale Characterizations of Fractal Dimensions). *Let $X \subseteq \mathbf{C}$.*

1 (Lutz [32]). $\dim_{\mathrm{H}}(X) = \inf \mathcal{G}(X)$.

2 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). $\dim_{\mathrm{P}}(X) = \inf \mathcal{G}^{\mathrm{str}}(X)$.

Informally speaking, the above theorem says that the dimension of a set is the *most hostile environment* (i.e., the most unfavorable payoff schedule, i.e., the infimum s) in which a single betting strategy can *achieve infinite winnings* on every element of the set. The two dimensions differ only in that "achieve infinite winnings" refers to success in the case of $\dim_{\mathrm{H}}$ and to strong success in the case of $\dim_{\mathrm{P}}$.

By Observation 2.1, we could equivalently use $\widehat{\mathcal{G}}(X)$ and $\widehat{\mathcal{G}}^{\mathrm{str}}(X)$ in Theorem 2.2.

The following obvious but useful fact shows how gales and supergales are affected by variation of the parameter s.

Observation 2.3 (Lutz [33]). *Let $s, s' \in [0, \infty)$, and let $d, d' : \{0,1\}^* \to [0, \infty)$. Assume that*

$$d(w)2^{-s|w|} = d'(w)2^{-s'|w|}$$

for all $w \in \{0,1\}^*$. Then d is an s-gale if and only if d' is an s'-gale, and similarly for supergales.

For example, Observation 2.3 implies that a function $d : \{0,1\}^* \to [0, \infty)$ is an s-gale if and only if the function $d' : \{0,1\}^* \to [0, \infty)$ defined by $d'(w) = 2^{(1-s)|w|}d(w)$ is a martingale. We can thus equivalently characterize the fractal dimensions $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ in terms of the *highest rate* at which a single betting strategy (now a martingale) can win in a *fair environment*.

Well known (and easily, derived) properties of the fractal dimensions $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ include the following. Each of the dimensions is *monotone*, i.e.,

$$X \subseteq Y \Rightarrow \dim(X) \leq \dim(Y),$$

and *countably stable*, i.e.,

$$\dim(\bigcup_{n=0}^{\infty} X_n) = \sup_{n \in \mathbb{N}} \dim(X_n).$$

For every set $X \subseteq \mathbf{C}$, we have

$$0 \leq \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{P}}(X) \leq 1.$$

If X is countable, then
$$\dim_{\mathrm{H}}(X) = \dim_{\mathrm{P}}(X) = 0,$$
while
$$\dim_{\mathrm{H}}(\mathbf{C}) = \dim_{\mathrm{P}}(\mathbf{C}) = 1.$$

3   Constructive Fractal Dimensions

Our first effectivization of fractal dimensions is at the *constructive* level.

Definition. An *s-supergale* d is *constructive* if it is lower semicomputable, i.e., if there is a computable function $\hat{d} : \{0, 1\}^* \times \mathbb{N} \longrightarrow \mathbb{Q}$ such that

(a) for all w, t, $\hat{d}(w, t) \leq \hat{d}(w, t + 1) < d(w)$, and

(b) for all w, $\lim_{t \to \infty} \hat{d}(w, t) = d(w)$.

Martin-Löf [39] used constructive measure theory to give the first satisfactory definition of the randomness of individual infinite binary sequences. This definition says precisely which infinite binary sequences are random and which are not. The definition is probabilistically convincing in that it requires each random sequence to pass every algorithmically implementable statistical test of randomness. The definition is also robust in that subsequent definitions by Schnorr [48, 49, 50], Levin [28], Chaitin [8], Solovay [55], and Shen′ [53, 54], using a variety of different approaches, all define exactly the same sequences to be random. In fact it is most useful to regard the following characterization as the definition of randomness for our purposes here.

Theorem 3.1 (Schnorr [48]). *A sequence $R \in \mathbf{C}$ is random if there is no constructive martingale that succeeds on R.*

We now define constructive versions of Hausdorff and packing dimension by requiring the gales in the characterizations of Theorem 2.2 to be constructive.

Notation. For $X \subseteq \mathbf{C}$, we define the sets $\mathcal{G}_{\mathrm{constr}}(X)$, $\mathcal{G}^{\mathrm{str}}_{\mathrm{constr}}(X)$, $\widehat{\mathcal{G}}_{\mathrm{constr}}(X)$, and $\widehat{\mathcal{G}}^{\mathrm{str}}_{\mathrm{constr}}(X)$ just as the classes $\mathcal{G}(X)$, $\mathcal{G}^{\mathrm{str}}(X)$, $\widehat{\mathcal{G}}(X)$, and $\widehat{\mathcal{G}}^{\mathrm{str}}(X)$ were defined in section 2, but with d now required to be constructive.

We do not know whether the constructive analog of Observation 2.1 holds, but the following theorem, which was proven independently by Hitchcock and Fenner, is sufficient for our purposes.

Theorem 3.2 (Hitchcock [23], Fenner [15]). *For all $X \subseteq \mathbf{C}$,*

$$\inf \mathcal{G}_{\mathrm{constr}} = \inf \widehat{\mathcal{G}}_{\mathrm{constr}}$$

and

$$\inf \mathcal{G}^{\mathrm{str}}_{\mathrm{constr}} = \inf \widehat{\mathcal{G}}^{\mathrm{str}}_{\mathrm{constr}}.$$

The constructive fractal dimensions are now defined as follows.

Definition. Let $X \subseteq \mathbf{C}$.

1 (Lutz [33]). The *constructive dimension* of X is $\mathrm{cdim}(X) = \inf \mathcal{G}_{\mathrm{constr}}(X)$.

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2 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). The *constructive strong dimension* of X is $\mathrm{cDim}(X) = \inf \mathcal{G}_{\mathrm{constr}}^{\mathrm{str}}(X)$.

A brief remark on notation is appropriate here. In the fractal geometry literature, there are two commonly used conventions for denoting the Hausdorff and packing dimensions of a set X. One is to denote them by $\dim_{\mathrm{H}}(X)$ and $\dim_{\mathrm{P}}(X)$, respectively, as in section 2 above. The other is to denote them by $\dim(X)$ and $\mathrm{Dim}(X)$, respectively, thereby avoiding subscripts. In denoting effective versions of Hausdorff and packing dimensions, we follow the latter convention, as in the use of $\mathrm{cdim}(X)$ and $\mathrm{cDim}(X)$ above.

Observation 3.3. *For every set $X \subseteq \mathbf{C}$,*

$$0 \leq \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{P}}(X)$$
$$\begin{array}{ccc} & |\wedge & \quad |\wedge \end{array}$$
$$\mathrm{cdim}(X) \;\leq\; \mathrm{cDim}X \;\leq 1.$$

As noted in section 2, every countable set of sequences has Hausdorff and packing dimension 0. In contrast, even a singleton set, consisting of a single sequence, may have positive constructive dimension, and this leads to the following useful definition.

Definition. Let $S \in \mathbf{C}$.

1 (Lutz [33]). The *dimension* of S is $\dim(S) = \mathrm{cdim}(\{s\})$.

2 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). The *strong dimension* of S is $\mathrm{Dim}(S) = \mathrm{cDim}(\{S\})$.

Although $\dim(S)$ and $\mathrm{Dim}(S)$ are constructive notions, it is convenient to omit "constructive" from the terminology and notation.

The following theorem, which has no analog either in classical fractal dimension or in the other effective fractal dimensions surveyed in this paper, says that the constructive dimension and strong dimension of a set of sequences are completely determined by the dimensions and strong dimensions of the individual sequences in the set.

Theorem 3.4. *Let $X \subseteq \mathbf{C}$.*

1 *(Lutz [33]).* $\mathrm{cdim}(X) = \sup_{S \in X} \dim(S)$.

2 *(Athreya, Hitchcock, Lutz, and Mayordomo [3]).* $\mathrm{cDim}(X) = \sup_{S \in X} \mathrm{Dim}(S)$.

The above theorem says that the constructive fractal dimensions are *absolutely stable* in the sense that for *every* indexed family $\{X_i \mid i \in \mathcal{I}\}$ of sets $X_i \subseteq \mathbf{C}$,

$$\mathrm{cdim}(\bigcup_{i \in \mathcal{I}} X_i) = \sup_{i \in \mathcal{I}} \mathrm{cdim}(X_i)$$

and

$$\mathrm{cDim}(\bigcup_{i \in \mathcal{I}} X_i) = \sup_{i \in \mathcal{I}} \mathrm{cDim}(X_i).$$

This is much stronger than the countable stability property of the classical fractal dimensions mentioned in section 2.

The following *correspondence principle* says that, for "simple" sets X, the constructive dimension coincides with the classical Hausdorff dimension.

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Theorem 3.5 (Hitchcock [20]). *If X is an arbitrary union of Π_1^0 (i.e., computably closed) subsets of $\mathbf{C}$, then*

$$\mathrm{cdim}(X) = \dim_{\mathrm{H}}(X).$$

An interesting open problem is to establish a similar correspondence principle for constructive strong dimension and packing dimension. In any case, Theorems 3.4 and 3.5 give the following *pointwise characterization* of the classical Hausdorff dimensions of "simple" sets X.

Corollary 3.6. *If X is an arbitrary union of Π_1^0 subsets of $\mathbf{C}$, then*

$$\dim_{\mathrm{H}}(X) = \sup_{S \in X} \dim(S).$$

It is clear from the definitions that every random sequence R has dimension 1. To further explore the existence and nature of sequences of various dimensions, we define, for each $\alpha \in [0, 1]$, the *level sets*

$$\mathrm{DIM}^{\alpha} = \{S \in \mathbf{C} | \dim(S) = \alpha\},$$

$$\mathrm{DIM}_{\mathrm{str}}^{\alpha} = \{S \in \mathbf{C} | \mathrm{Dim}(S) = \alpha\}.$$

We also use the notations $\mathrm{DIM}^{<\alpha}$, $\mathrm{DIM}^{\leq\alpha}$, etc., with the obvious meanings.

Theorem 3.7. *Let $\alpha \in [0, 1]$.*

1. *(Lutz [33]).* $\mathrm{cdim}(\mathrm{DIM}^{\alpha}) = \mathrm{cdim}(\mathrm{DIM}^{<\alpha}) = \dim_{\mathrm{H}}(\mathrm{DIM}^{\alpha}) = \dim_{\mathrm{H}}(\mathrm{DIM}^{<\alpha}) = \alpha$.

2. $\mathrm{cDim}(\mathrm{DIM}_{\mathrm{str}}^{\alpha}) = \mathrm{cDim}(\mathrm{DIM}_{\mathrm{str}}^{\leq\alpha}) = \dim_{\mathrm{P}}(\mathrm{DIM}_{\mathrm{str}}^{\alpha}) = \dim_{\mathrm{P}}(\mathrm{DIM}_{\mathrm{str}}^{\leq\alpha}) = \alpha$.

The above theorem implies the existence of sequences with any given dimension or strong dimension. The following theorem establishes the existence of sequences with any given dimension *and* strong dimension.

Theorem 3.8 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). *For any two real numbers $0 \leq \alpha \leq \beta \leq 1$, there is a sequence $S \in \mathbf{C}$ such that $\dim(S) = \alpha$ and $\mathrm{Dim}(S) = \beta$.*

What might such a sequence look like, and how simple can it be? to answer this question, we recall the generalization of randomness to an arbitrary probability measure on $\mathbf{C}$.

A *probability measure* on $\mathbf{C}$ is a function $\nu : \{0, 1\}^* \to [0, \infty)$ such that $\nu(\lambda) = 1$ and $\nu(w) = \nu(w0) + \nu(w1)$ for all $w \in \{0, 1\}^*$. (Intuitively, $\nu(w)$ is the probability that $w \sqsubseteq S$ when the sequence S is "chosen according to ν.")

A *bias* is a real number $\beta \in [0, 1]$. Intuitively, if we toss a 0/1-valued coin with bias β, then β is the probability of the outcome 1. A *bias sequence* is a sequence $\vec{\beta} = (\beta_0, \beta_1, \beta_2, \ldots)$ of biases. If $\vec{\beta}$ is a bias sequence, then the *$\vec{\beta}$-coin-toss probability measure* is the probability $\mu^{\vec{\beta}}$ on $\mathbf{C}$ defined by

$$\mu^{\vec{\beta}}(w) = \prod_{i=0}^{|w|-1} \beta_i(w), \tag{3.1}$$

where $\beta_i(w) = (2\beta_i - 1)w[i] + (1 - \beta_i)$, i.e., $\beta_i(w) = \mathbf{if}\ w[i]\ \mathbf{then}\ \beta_i\ \mathbf{else}\ 1 - \beta_i$. That is, $\mu^{\vec{\beta}}$ is the probability that $S \in \mathbf{C}_w$ when $S \in \mathbf{C}$ is chosen according to a random experiment in which for each i, independently of all other j, the ith bit of S is decided by tossing a 0/1-valued coin whose

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probability of 1 is β_i. In the case where the biases β_i are all the same, i.e., $\vec{\beta} = (\beta, \beta, \beta, \ldots)$ for some $\beta \in [0, 1]$, we write μ^β for $\mu^{\vec{\beta}}$, and (3.1) simplifies to

$$\mu^\beta(w) = (1 - \beta)^{\#(0,w)} \beta^{\#(1,w)}, \tag{3.2}$$

where $\#(b, w)$ is the number of times the bit b appears in the string w. The *uniform probability measure* on $\mathbf{C}$ is the probability measure $\mu = \mu^{\frac{1}{2}}$, for which (3.2) simplifies to

$$\mu(w) = 2^{-|w|}$$

for all $w \in \{0, 1\}^*$.

Definition. Let ν be a probability measure on $\mathbf{C}$.

1. A ν-*martingale* is a function $d : \{0, 1\}^* \to [0, \infty)$ that satisfies the condition

$$d(w)\nu(w) = d(w0)\nu(w0) + d(w1)\nu(w1)$$

   for all $w \in \{0, 1\}^*$.

2. A ν-martingale is *constructive* if it is lower semicomputable.

Note that a μ-martingale is a martingale. If $\vec{\beta}$ is a bias sequence, then we call a $\mu^{\vec{\beta}}$-martingale simply a $\vec{\beta}$-martingale.

Definition (Schnorr [48]). If ν is a probability measure on $\mathbf{C}$, then a sequence $R \in \mathbf{C}$ is ν-*random* if there is no constructive ν-martingale that succeeds on R.

Given a bias sequence $\vec{\beta}$, we say that a sequence $R \in \mathbf{C}$ is $\vec{\beta}$-*random* if it is $\mu^{\vec{\beta}}$-random. Note that a sequence is random if and only if it is $\vec{\beta}$-random, where $\vec{\beta} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$.

Recall that the *Shannon entropy* of a bias $\beta \in [0, 1]$ is

$$\mathcal{H}(\beta) = \beta \log \frac{1}{\beta} + (1 - \beta) \log \frac{1}{1 - \beta},$$

where we insist that $0 \log \frac{1}{0} = 0$.

Notation. Given a bias sequence $\vec{\beta} = (\beta_0, \beta_1, \ldots)$, $n \in \mathbb{N}$, and $S \in \mathbf{C}$, let

$$
\begin{aligned}
H_n(\vec{\beta}) &= \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{H}(\beta_i), \\
H^-(\vec{\beta}) &= \liminf_{n \to \infty} H_n(\vec{\beta}), \\
H^+(\vec{\beta}) &= \limsup_{n \to \infty} H_n(\vec{\beta}).
\end{aligned}
$$

We call $H^-(\vec{\beta})$ and $H^+(\vec{\beta})$ the *lower* and *upper average entropies*, respectively, of $\vec{\beta}$.

Theorem 3.9 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). *If $\delta \in (0, \frac{1}{2}]$ and $\vec{\beta}$ is a computable bias sequence with each $\beta_i \in [\delta, \frac{1}{2}]$, then for every sequence $R \in \text{RAND}^{\vec{\beta}}$,*

$$\dim(R) = H^-(\vec{\beta}) \text{ and } \operatorname{Dim}(R) = H^+(\vec{\beta}).$$

Theorem 3.9 says that every sequence that is random with respect to a suitable bias sequence $\vec{\beta}$ has the lower and upper average entropies of $\vec{\beta}$ as its dimension and strong dimension, respectively. Since there exist $\vec{\beta}$-random sequences in Δ_2^0 when $\vec{\beta}$ is computable, this gives a powerful and flexible method for constructing Δ_2^0 sequences with given (Δ_2^0-computable) dimensions and strong dimensions.

Let Σ_1^0 be the set of sequences S for which $\{n|S[n]=1\}$ is computably enumerable, and let Π_1^0 be the set of sequences S for which $\{n|S[n]=0\}$ is computably enumerable.

Theorem 3.10 (Lutz [33]). $\Sigma_1^0 \cup \Pi_1^0 \subseteq \mathrm{DIM}_{\mathrm{str}}^0$. *(This was actually proven in [33], though stated in the weaker form $\Sigma_1^0 \cup \Pi_1^0 \subseteq \mathrm{DIM}^0$.)*

By Theorems 3.9 and 3.10, sequences of positive dimension or strong dimension occur in Δ_2^0, but no lower, in the arithmetical hierarchy.

We conclude this section by considering the arithmetical complexities of the level sets DIM^α.

Theorem 3.11 (Hitchcock, Lutz, and Terwijn [26]).   *1. The level set DIM^0 is Π_2^0.*

  2. If $\alpha \in (0,1]$ is Δ_2^0-computable, then the level set DIM^α is Π_3^0, but not Σ_3^0.

In contrast with DIM^1, it is well known that the set of random sequences is Π_2^0.

The complexities of the level sets $\mathrm{DIM}_{\mathrm{str}}^\alpha$ are also considered in [26], but the statement of that result is a bit more involved.

4   Discrete Dimension and Kolmogorov Complexity

We have now seen that the classical fractal dimensions can be constructivized, thereby defining the dimensions and strong dimensions of individual infinite binary sequences. In [33] we pushed this a step further by constructivizing *and discretizing* classical fractal dimension in order to define the dimensions of individual *finite* binary strings.

Recall that the dimension of a sequence S is the infimum of all $s \geq 0$ for which there exists a constructive s-supergale d such that the values of $d(S[0..n-1])$ are unbounded as $n \to \infty$. To define the dimensions of finite strings, we modify this definition in three ways.

  I. We replace gales by termgales, which are gale-like constructs with special requirements for handling the terminations of strings.

 II. We replace "unbounded as $n \to \infty$" by a finite threshold.

III. We make the definition universal by using an optimal constructive termgale.

We refer the reader to [33] for the details of this development. The result is that each binary string x is assigned a *discrete dimension* $\dim(X)$ that "agrees" with the dimension and strong dimensions in the following asymptotic sense.

Theorem 4.1. *Let $S \in \mathbf{C}$.*

1 (Lutz [33]). $\dim(S) = \liminf_{n\to\infty} \dim(S[0..n-1])$.

2 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). $\mathrm{Dim}(S) = \limsup_{n\to\infty} \dim(S[0..n-1])$.

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It turns out that discrete dimension gives a new characterization of Kolmogorov complexity. (See the text by Li and Vitányi [31] for a thorough treatment of Kolmogorov complexity.)

Theorem 4.2 (Lutz [33]). *There is a constant $c \in \mathbb{N}$ such that, for all $x \in \{0,1\}^*$,*

$$\left| K(x) - |x|\dim(x) \right| \leq c.$$

That is, the Kolmogorov complexity of a string is (to within a constant additive term) the product of the string's length and its dimension. This characterization of Kolmogorov complexity in terms of a constructivized, discretized version of Hausdorff's 1919 theory of dimension is reminiscent of (and technically related to) the well-known characterization by Levin [28, 29] and Chaitin [8] of Kolmogorov complexity in terms of constructivized discrete probability, i.e., the fact that there is a constant $c' \in \mathbb{N}$ such that for all $x \in \{0,1\}^*$,

$$\left| K(x) - \log \frac{1}{\mathbf{m}(x)} \right| \leq c',$$

where $\mathbf{m}$ is an optimal constructive subprobability measure on $\{0,1\}^*$.

The following theorem is an immediate consequence of Theorems 4.1 and 4.2.

Theorem 4.3. *Let $S \in \mathbf{C}$.*

1 (Mayordomo [42]). $\dim(S) = \liminf_{n \to \infty} \frac{K(S[0..n-1])}{n}$.

2 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). $\text{Dim}(S) = \limsup_{n \to \infty} \frac{K(S[0..n-1])}{n}$.

Mayordomo's proof of part 1 actually preceeded the formulation of discrete dimension. Several proofs of this result are now known. Earlier results by Ryabko [44, 45, 46, 47], Staiger [57, 58, 56], and Cai and Hartmanis [7] that relate martingales, supermartingales, and Kolmogorov complexity to Hausdorff dimension (using fundamental work by Levin [65] and Schnorr [49, 51]) are discussed by Lutz [33] and Staiger [59].

Theorems 4.2 and 4.3 justify the intuition that $\dim(x)$, $\dim(S)$, and $\text{Dim}(S)$ are measures of *information density.*

Generalizing the construction of Chaitin's random real number Ω [8], Mayordomo [42] and, independently, Tadaki [61] defined for each $s \in (0,1]$ and each infinite, computably enumerable set $A \subseteq \{0,1\}^*$, the real number

$$\theta_A^s = \sum \left\{ 2^{\frac{|\pi|}{s}} \,\middle|\, \pi \in \{0,1\}^* \text{ and } U(\pi) \in A \right\},$$

where U is a universal self-delimiting Turing machine. Given Theorem 4.3, the following fact is implicit in Tadaki's paper.

Theorem 4.4. (Tadaki [61]) *For each $s \in (0,1]$ and each infinite, computably enumerable set $A \subseteq \{0,1\}^*$, the (binary expansion of the) real number θ_A^s satisfies $\dim(\theta_A^s) = \text{Dim}(\theta_A^s) = s$.*

5   Dimensions in Complexity Classes

One of the main reasons for effectivizing fractal dimensions is to impose useful internal dimension structure on various complexity classes. This structure is a refinement of the internal measure that resource-bounded measure (developed by Lutz [34, 36], surveyed by Lutz [35], Ambos-Spies and Mayordomo [1], Buhrman and Torenvliet [5], and Lutz and Mayordomo [38], and extensively documented by Hitchcock's bibliography [19]) imposes on these classes.

A *language*, or *decision problem*, is a set $A \subseteq \{0,1\}^*$. We usually identify a language A with its characteristic sequence $\chi_A \in \mathbf{C}$ defined by $\chi_A[n] = \mathbf{if}\ s_n \in A\ \mathbf{then}\ 1\ \mathbf{else}\ 0$, where $s_0 = \lambda, s_1 = 0, s_2 = 1, s_3 = 00, \ldots$ is the standard enumeration of $\{0,1\}^*$. That is, we usually (but not always) use A to denote both the set $A \subseteq \{0,1\}^*$ and the sequence $A = \chi_A \in \mathbf{C}$.

We use the following classes of functions as *resource bounds*.

   all $= \{f \,|\, f : \{0,1\}^* \to \{0,1\}^*\}$
   comp $= \{f \in$ all $| f$ is computable $\}$
   p $= \{f \in$ all $| f$ is computable in polynomial time $\}$
   $\mathrm{p}_2 = \{f \in$ all $| f$ is computable in $n^{(\log n)^{O(1)}}$ time $\}$
   pspace $= \{f \in$ all $| f$ is computable in polynomial space $\}$
   $\mathrm{p}_2$space $= \{f \in$ all $| f$ is computable in $n^{(\log n)^{O(1)}}$ space $\}$

A *constructor* is a function $\delta : \{0,1\}^* \to \{0,1\}^*$ that satisfies $x \underset{\neq}{\sqsubset} \delta(x)$ for all x. The *result* of a constructor δ (i.e., the language *constructed* by δ) is the unique language $R(\delta)$ such that $\delta^n(\lambda) \sqsubseteq R(\delta)$ for all $n \in \mathbb{N}$. Intuitively, δ constructs $R(\delta)$ by starting with λ and then iteratively generating successively longer prefixes of $R(\delta)$. We write $R(\Delta)$ for the set of languages $R(\delta)$ such that δ is a constructor in Δ. The following facts are the reason for our interest in the above-defined classes of functions.

   $R($all$) = \mathbf{C}$.
   $R($comp$) = $ DEC.
   $R($p$) = $ E $= $ DTIME(2^{linear}).
   $R(\mathrm{p}_2) = $ EXP $= $ DTIME$(2^{\mathrm{polynomial}})$.
   $R($pspace$) = $ ESPACE $= $ DSPACE(2^{linear}).
   $R(\mathrm{p}_2$space$) = $ EXPSPACE $= $ DSPACE$(2^{\mathrm{polynomial}})$.

Throughout this section, Δ denotes one of the resource bounds all, comp, p, p_2, pspace, $\mathrm{p}_2$space defined above. An s-supergale d is Δ-*computable* if there is a function $\hat{d} : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q} \cap [0, \infty)$ such that $|\hat{d}(w,r) - d(w)| \leq 2^{-r}$ for all $w \in \{0,1\}^*$ and $r \in \mathbb{N}$ and $\hat{d} \in \Delta$ (with r coded in unary and the output coded in binary). For $X \subseteq \mathbf{C}$, we then define the classes $\mathcal{G}_\Delta(X)$, $\mathcal{G}_\Delta^{\mathrm{str}}(X)$, $\widehat{\mathcal{G}}_\Delta(X)$, and $\widehat{\mathcal{G}}_\Delta^{\mathrm{str}}(X)$ just as the classes $\mathcal{G}(X)$, $\mathcal{G}^{\mathrm{str}}(X)$, $\widehat{\mathcal{G}}(X)$, and $\widehat{\mathcal{G}}^{\mathrm{str}}(X)$ were defined in section 2, but with d now required to be computable.

Definition (Lutz [32], Athreya, Hitchcock, Lutz, and Mayordomo [3]). Let $X \subseteq \mathbf{C}$.

1. The Δ-*dimension* of X is
$$\dim_\Delta(X) = \inf \mathcal{G}_\Delta(X).$$

2. The Δ-*strong dimension* of X is
$$\mathrm{Dim}_\Delta(X) = \inf \mathcal{G}_\Delta^{\mathrm{str}}(X).$$

3. The *dimension* of X *in* $R(\Delta)$ is
$$\dim(X | R(\Delta)) = \dim_\Delta(X \cap R(\Delta)).$$

4. The *strong dimension* of X *in* $R(\Delta)$ is

$$\text{Dim}(X|R(\Delta)) = \text{Dim}_\Delta(X \cap R(\Delta)).$$

In parts 1 and 2 of the above definition, we could equivalently use the "hatted" sets $\widehat{\mathcal{G}}_\Delta(X)$ and $\widehat{\mathcal{G}}_\Delta^{\text{str}}(X)$ in place of their unhatted counterparts.

The polynomial-time dimensions $\text{dim}_p(X)$ and $\text{Dim}_p(X)$ are also called the *feasible dimension* and the *feasible strong dimension*, respectively. The notation $\text{dim}_p(X)$ for the p-dimension is all too similar to the notation $\text{dim}_P(X)$ for the classical packing dimension, but confusion is unlikely because these dimensions typically arise in quite different contexts.

Note that the classical Hausdorff and packing dimensions can each now be written in three different ways, i.e.,

$$\text{dim}_H(X) = \text{dim}_{\text{all}}(X) = \text{dim}(X|\mathbf{C})$$

and

$$\text{dim}_P(X) = \text{Dim}_{\text{all}}(X) = \text{Dim}(X|\mathbf{C}).$$

We are, of course, more interested in the *effective* fractal dimensions $\text{dim}_\Delta(X)$ and $\text{Dim}_\Delta(X)$, where $\Delta \subseteq \text{comp}$, and the structures they impose on the corresponding classes $R(\Delta)$. A critical fact about these structures is that they do not "collapse" the classes $R(\Delta)$, even though the latter is countable where $\Delta \subseteq \text{comp}$.

Theorem 5.1 (Lutz [32]). $\text{dim}(R(\Delta)|R(\Delta)) = \text{dim}_\Delta(R(\Delta)) = 1$.

In fact, resource-bounded dimension *refines* resource-bounded measure in the sense that

$$\text{dim}_\Delta(X) < 1 \Rightarrow \mu_\Delta(X) = 0$$

and

$$\text{dim}(X|R(\Delta)) < 1 \Rightarrow \mu(X|R(\Delta)) = 0.$$

Thus, for example, if $\text{dim}(X|\text{E}) < 1$, then X has measure 0 in E, i.e., $X \cap \text{E}$ is a negligibly small subset of E.

We now consider some particular complexity-theoretic topics from the standpoint of dimension.

For each $s : \mathbb{N} \to \mathbb{N}$, let $\text{SIZE}(s(n))$ be the class of all languages $A \subseteq \{0, 1\}^*$ such that, for each $n \in \mathbb{N}$, $A_{=n}$ is decided by a Boolean circuit consisting of at most $s(n)$ gates. Shannon [52] showed (essentially) that $\text{SIZE}(\alpha \frac{2^n}{n})$ has measure 0 in $\mathbf{C}$ for all $\alpha < 1$, and Lutz [34] showed that $\text{SIZE}(\alpha \frac{2^n}{n})$ also has measure 0 in ESPACE for all $\alpha < 1$. We now use resource-bounded dimension to give a quantitative refinement of these results.

Theorem 5.2 (Lutz [32], Athreya, Hitchcock, Lutz, and Mayordomo [3]). *For each* $\alpha \in [0, 1]$, *the class* $X_\alpha = \text{SIZE}(\alpha \cdot \frac{2^n}{n})$ *satisfies* $\text{dim}_{\text{pspace}}(X_\alpha) = \text{Dim}_{\text{pspace}}(X_\alpha) = \text{dim}(X_\alpha|\text{ESPACE}) = \text{Dim}(X_\alpha|\text{ESPACE}) = \alpha$.

It was shown by Juedes and Lutz [27] that every polynomial-time many-one degree has measure 0 in E. The following refinement of this result showed that the dimensions of such degrees are unrestricted in E.

Theorem 5.3 (Ambos-Spies, Merkle, Reimann, and Stephan [2]). *For every* Δ_2^0*-computable real number* $x \in [0, 1]$, *there exists* $A \in \text{E}$ *such that*

$$\text{dim}_p(\text{deg}_m^p(A)) = \text{dim}(\text{deg}_m^p(A)|\text{E}) = x.$$

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This result was recently extended to the following.

Theorem 5.4 (Athreya, Hitchcock, Lutz, and Mayordomo [3]). *For every pair of Δ_2^0-computable real numbers x, y with $0 \le x \le y \le 1$, there exists $A \in E$ such that*

$$\dim_{\mathrm{p}}(\deg_{\mathrm{m}}^{\mathrm{p}}(A)) = \dim(\deg_{\mathrm{m}}^{\mathrm{p}}(A)|E) = x$$

and

$$\mathrm{Dim}_{\mathrm{p}}(\deg_{\mathrm{m}}^{\mathrm{p}}(A)) = \mathrm{Dim}(\deg_{\mathrm{m}}^{\mathrm{p}}(A)|E) = y.$$

We note that the proofs of Theorems 5.3 and 5.4 - especially the latter - are not straightforward and involve an unusual variety of techniques.

The hypothesis $\mu_{\mathrm{p}}(\mathrm{NP}) \ne 0$ ("NP does not have p-measure 0") is known to have many complexity-theoretic consequences not known to follow from more "traditional" hypotheses such as $\mathrm{P} \ne \mathrm{NP}$ or the separation of the polynomial-time hierarchy into infinitely many levels. The hypothesis $\dim_{\mathrm{p}}(\mathrm{NP}) > 0$ is ostensibly weaker, but still very strong because

$$\mu_{\mathrm{p}}(\mathrm{NP}) \ne 0 \Rightarrow \dim_{\mathrm{p}}(\mathrm{NP}) = 1 \Rightarrow \dim_{\mathrm{p}}(\mathrm{NP}) > 0 \Rightarrow \mathrm{P} \ne \mathrm{NP}.$$

In fact, the hypothesis $\dim_{\mathrm{p}}(\mathrm{NP}) > 0$ is now known to have the following consequence for the difficulty of approximating MAX3SAT.

Theorem 5.5 (Hitchcock [21]). *If $\dim_{\mathrm{p}}(\mathrm{NP}) > 0$, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that any $2^{n^{\delta}}$-time approximation algorithm for MAX3SAT has performance ratio less than $\frac{7}{8} + \epsilon$ on a dense set of satisfiable instances.*

A language A is *autoreducible* if there is a polynomial-time Turing reduction of A to itself that never queries the oracle on its input. The language A is *infinitely often autoreducible* if there is a polynomial-time oracle machine M that never queries the oracle on its input and, on oracle A, satisfies the following two conditions.

(i) For infinitely many inputs x, $M^A(x)$ correctly decides whether $x \in A$.

(ii) For all other x, $M^A(x)$ outputs a special "undefined" symbol.

Let AR be the class of autoreducible languages, and let $AR^{\mathrm{i.o.}}$ be the class of infinitely often autoreducible languages. The measure of AR in EXP is unknown and has bearing on the BPP versus EXP problem [6]. The following dimension result at least puts a limit on how "small" AR can be in EXP.

Observation 5.6 (Ambos-Spies, Merkle, Reimann, and Stephan [2]). $\dim(AR|\mathrm{EXP}) = 1$.

In a recent celebrated result, Ebert [11, 12] proved that $AR^{\mathrm{i.o.}}$ has measure 1 in EXP, i.e., almost every language in EXP is infinitely often autoreducible. Thus the complement $\mathrm{EXP} - AR^{\mathrm{i.o.}}$ has measure 0 in EXP. The following result shows that this set, too, has dimension 1 in EXP.

Theorem 5.7 (Beigel, Fortnow, and Stephan [4]). $\dim(\mathrm{EXP} - AR^{\mathrm{i.o.}}|\mathrm{EXP}) = 1$.

6 Other Work

Effective fractal dimension is a very new research area, but it is growing rapidly, and there are several developments that have not been discussed. We briefly mention a few of these here.

Many classes that occur naturally in computational complexity are parametrized in such a way as to remain out of rearch of the resource-bounded dimensions defined in section 5. Hitchcock, Lutz, and Mayordomo [25] have thus extended the resource-bounded dimension of section 5 by introducing the notion of a *scale* according to which dimension may be measured. These scales are slightly less general than the functions used for classical generalized dimension [17, 43] and take two arguments instead of one, but every scale g defines for every set X of decision problems a *g-scaled dimension* $\dim^g(X) \in [0, 1]$. The choice of which scale to use for a particular application is very much like the choice of whether to plot data on a standard Cartesian graph or a log-log graph. In fact, a very restricted family of scales appears to be adequate for analyzing many problems in computational complexity. These scales are used in [25] to investigate Boolean circuit-size complexity and resource-bounded Kolmogorov complexity. Hitchcock [24] uses scaled dimension to investigate "small span" phenomena.

Fortnow and Lutz [16] gave precise quantitative bounds on the relationship between feasible dimension and feasible predictability in the absolute loss model. Hitchcock [22] proved that, at any level from feasible to classical, fractal dimension is precisely unpredictability in the logarithmic loss model. This characterization has already been useful in proving some of the results surveyed here.

Dai, Lathrop, Lutz, and Mayordomo [10] have formulated *finite-state-dimension* and used it to characterize finte-state compressibility. A dual characterization has been noted in [3].

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