Effective Dimensions and Relative Frequencies

Xiaoyang Gu[†] and Jack H. Lutz[†‡]

Department of Computer Science, Iowa State University, Ames, IA 50011 USA.
{xiaoyang,lutz}@cs.iastate.edu.

Abstract. Consider the problem of calculating the fractal dimension of a set X consisting of all infinite sequences S over a finite alphabet Σ that satisfy some given condition P on the asymptotic frequencies with which various symbols from Σ appear in S. Solutions to this problem are known in cases where

(i) the fractal dimension is classical (Hausdorff or packing dimension), or

(ii) the fractal dimension is effective (even finite-state) and the condition P *completely* specifies an empirical distribution π over Σ, i.e., a limiting frequency of occurrence for *every* symbol in Σ.

In this paper we show how to calculate the finite-state dimension (equivalently, the finite-state compressibility) of such a set X when the condition P only imposes *partial* constraints on the limiting frequencies of symbols. Our results automatically extend to less restrictive effective fractal dimensions (e.g., polynomial-time, computable, and constructive dimensions), and they have the classical results (i) as immediate corollaries. Our methods are nevertheless elementary and, in most cases, simpler than those by which the classical results were obtained.

Keywords: effective fractal dimensions, empirical frequencies, finite-state dimension, randomness, saturated sets

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[‡] corresponding author.

1 Introduction

The most fundamental statistics used in the analysis of data for purposes of compression or prediction are the empirical frequencies with which various symbols appear. When *every* symbol has a frequency that is known and stable throughout the data, the problems of compression and prediction are well understood, with the main insights now over a half-century old [15, 16, 5]. However, when only *partial* constraints on the empirical frequencies–e.g., the *relative* frequencies of some of the symbols–are known, these problems become more challenging.

This paper shows how to calculate the finite-state dimension (equivalently, the compressibility or predictability by finite-state machines [6, 14]) of a set X of infinite sequences over a finite alphabet Σ when membership of a sequence S in X is determined by some given condition P on the asymptotic frequencies with which various symbols from Σ appear in S. Our results hold even when P only imposes partial constraints on the limiting frequencies of symbols, and they automatically extend to less restrictive effective dimensions, such as polynomial-time, computable, and constructive dimensions. In order to explain our results and their significance, we briefly review four lines of research that are precursors of our work.

1.1 Classical fractal dimensions

In 1919, Hausdorff [13] developed a rigorous way of assigning a dimension to every subset of an arbitrary metric space. His definition agrees with the intuitive notion of dimension for "smooth" sets (e.g., smooth curves have dimension 1; smooth surfaces have dimension 2), but assigns non-integer dimensions to some more exotic sets, and hence came to be called a "fractal" dimension. In 1949, Eggleston [7], building on work of Besicovitch [3] and Good [10], proved that, for any probability measure π on a finite alphabet Σ, the set of all sequences in which each symbol $a \in \Sigma$ has asymptotic frequency $\pi(a)$ has Hausdorff dimension $\mathcal{H}_{|\Sigma|}(\pi)$, the Shannon entropy of π, normalized to range over $[0, 1]$. In retrospect, Hausdorff dimension is an information-theoretic concept [27], but these developments essentially all took place prior to Shannon's development of information theory [28].

In the early 1980's, another fractal dimension called packing dimension, was introduced [30, 29]. Packing dimension agrees with Hausdorff on "regular" sets, but is larger on some sets [9].

1.2 Shannon information theory

In 1948, Shannon [28] developed a probabilistic theory of information (Shannon entropy) that has been enormously productive and is the setting in which most work on compression and prediction has been carried out [5].

1.3 Effective fractal dimensions

In 2000, Lutz [18, 19] proved a new characterization of Hausdorff dimension in terms of betting strategies and used this characterization to formulate effective fractal dimensions ranging from polynomial-time and polynomial-space dimensions to computable and constructive dimensions. Pushing this effort further, finite-state dimension was introduced the following year [6], and is now known to characterize the compressibility [6] and predictability [14] of sequences over finite alphabets. In [1], packing dimension was shown to have a betting-strategy characterization that is exactly dual to that of Hausdorff dimension, thereby giving dual "strong dimensions" at each of the levels of effectivity for which dimensions had been defined. Each of the papers mentioned here extended the above-mentioned result of Eggleston to the effective dimension(s) introduced. Hence, as indicated in our first paragraph, the compression and prediction problems are well understood, even at the finite-state level, when a set X of sequences is defined in terms of given, well-defined, asymptotic frequencies of all symbols.

1.4 Classical dimensions of saturated sets

In 2002, Barreira, Saussol, and Schmeling [2] considered the classical fractal dimensions of sets of sequences defined in terms of conditions placing (typically partial) constraints on the frequencies and relative frequencies of symbols. The example by which they introduced their work was the set X of all sequences over the alphabet $\{0, 1, 2, 3\}$ in which there are asymptotically five times as many 0's as 1's. (No constraint is placed on the frequency of any individual symbol.) Using sophisticated techniques (multifractal analysis and ergodic theory), they showed how to compute the classical Hausdorff dimensions of sets of this kind. As it turns out, Volkmann [31] and his student Cajar [4] had previously defined a set X of sequences to be *saturated* if membership in it is completely determined by the asymptotic behaviors (not necessarily convergent) of the frequencies of symbols and investigated the Hausdorff dimensions of many saturated sets. Olsen [21–24] and Olsen and Winter [25, 26] also used multifractal analysis to study such sets.

1.5 Our results

We show how to calculate the finite-state dimensions of saturated sets. We give a pointwise characterization of the dimensions of such sets, and we prove a general correspondence principle stating that, if X is any saturated set, then the finite-state dimension of X is exactly its classical Hausdorff dimension, and the finite-state strong dimension of X is exactly its classical packing dimension. We also give completely elementary methods (no multifractal analysis or ergodic theory) for computing the finite-state dimensions of various types of saturated sets. By our correspondence principle, this yields elementary proofs that these results also hold for classical fractal dimensions and less restrictive effective fractal dimensions.

The rest of this paper is organized as follows. Section 2 lists the basic definitions and conventions we use in this paper. Section 3 reviews the definitions of Hausdorff dimension, packing dimension, finite-state dimension, and finite-state strong dimension. We give a few example of calculating the dimensions of exotic saturated sets in Section 4. In Section 5, we discuss finite-state dimensions of saturated sets in detail and give insight into why a maximum entropy principle holds.

2 Preliminaries

Let $m \geq 2$ be an integer. We work with the m-ary alphabet $\Sigma_m = \{0, 1, \ldots, m-1\}$. Σ_m^* is the set of all (finite) *strings* on Σ_m including the empty string λ. $\mathbf{C}_m = \Sigma_m^\infty$ is the of all (infinite) m-ary *sequences*. $\mathbf{C} = \mathbf{C}_2$ is the Cantor space. $\Delta(\Sigma_m)$ is the set of all probability measures on Σ_m.

Let i be an integer such that $0 \leq i \leq m-1$. The symbol counting function $\#_i : (\mathbf{C}_m \cup \Sigma_m^*) \times \mathbb{N} \to \mathbb{N}$ is defined such that for every string or sequence S and $n \in \mathbb{N}$, $\#_i(S, n)$ is the number of occurrences of i in the first n bits of S. The symbol frequency function $\pi_i : (\mathbf{C}_m \cup \Sigma_m^*) \times \mathbb{N} \to [0, 1]$ is defined such that $\pi_i(S, n) = \#_i(S, n)/n$. The empirical measure function $\vec{\pi} : (\mathbf{C}_m \cup \Sigma_m^*) \times \mathbb{N} \to \Delta(\Sigma_m)$ is defined such that $\vec{\pi}(S, n) = (\pi_0(S, n), \ldots, \pi_{m-1}(S, n))$. Intuitively, $\vec{\pi}$ extracts empirical probability measures from the first n bits of a string or a sequence based on the actual frequencies of digits.

3 The Four Dimensions

Hausdorff dimension and packing dimension are important tools in mathematics used to study the size of sets and the properties of dynamic systems. All countable sets have 0 for both of these dimensions. In order to study relative size of countable sets from the eyes of computers with different resources, Lutz generalized Hausdorff dimension to effective dimensions by using his gale characterization of Hausdorff dimension [18]. Athreya, Hitchcock, Lutz, and Mayordomo then gave a dual gale characterization of packing dimension, with which, they generalized packing dimension to effective strong dimensions [1]. We first review the definitions related to gales. Note that Σ_m is an alphabet with m symbols and $m \geq 2$.

Definition. Let $s \in [0, \infty)$. An *s-supergale* is a function $d : \Sigma_m^* \to [0, \infty)$ such that for all $w \in \Sigma_m^*$ $m^s d(w) \geq \sum_{a \in \Sigma_m} d(wa)$. The *success set* of an s-supergale d is $S^\infty[d] = \{S \in \mathbf{C} \mid \limsup_{n\to\infty} d(S[0..n-1]) = \infty\}$. The *strong success set* of d is $S^\infty_{\text{str}}[d] = \{S \in \mathbf{C} \mid \liminf_{n\to\infty} d(S[0..n-1]) = \infty\}$.

Now we conveniently give the gale characterizations of Hausdorff and packing dimensions as definitions. Please refer to Falconer [8] for classical definitions.

Definition. ([18, 1]). Let $X \subseteq \mathbf{C}_m$. The *Hausdorff dimension* of X is

$$\dim_{\mathrm{H}}(X) = \inf\{s \in [0,\infty) \mid X \subseteq S^{\infty}[d] \text{ for some } s\text{-supergale } d\ \}\,.$$

The *packing dimension* of X is

$$\dim_{\mathrm{P}}(X) = \inf\{s \in [0,\infty) \mid X \subseteq S^{\infty}_{\mathrm{str}}[d] \text{ for some } s\text{-supergale } d\}\,.$$

Finite-state dimension and strong dimension are finite-state counterparts of classical Hausdorff dimension [13] and packing dimension [20, 29] introduced by Dai, Lathrop, Lutz, and Mayordomo [6] and Athreya, Hitchcock, Lutz, and Mayordomo [1] in the Cantor space $\mathbf{C}$. Finite-state dimensions are defined by using the gale characterizations of the Hausdorff dimension [18] and the packing dimension [1] and restricting the gales to the ones whose underlying betting strategies can be carried out by finite-state gamblers. In this section, we give the definitions of the finite-state dimensions for space $\mathbf{C}_m$ and review their basic properties. Now, we define finite-state gamblers on alphabet Σ_m.

Definition. ([6]) A *finite-state gambler* (*FSG*) is a 5-tuple $G = (Q, \Sigma_m, \delta, \vec{\beta}, q_0)$ such that Q is a non-empty finite set of *states*; Σ_m is the input alphabet; $\delta : Q \times \Sigma_m \to Q$ is the *state transition function*; $\vec{\beta} : Q \to \Delta(\Sigma_m)$ is the *betting function*; $q_0 \in Q$ is the *initial state*.

The extended transition function $\delta^* : Q \times \Sigma_m^* \to Q$ is defined such that

$$\delta^*(q, wa) = \begin{cases} q & \text{if } w = a = \lambda, \\ \delta(\delta^*(q,w), a) & \text{if } w \neq \lambda. \end{cases}$$

We use δ for δ^* and $\delta(w)$ for $\delta(q_0, w)$ for convenience.

The betting function $\beta_i : Q \to \Delta(\Sigma_m)$ specifies the bets the FSG places on each input symbol in Σ_m with respect to a state $q \in Q$.

Definition. ([6]). Let $G = (Q, \Sigma_m, \delta, \vec{\beta}, q_0)$ be an FSG. The *s-gale* of G is the function $d_G : \Sigma_m^* \to [0,\infty)$ defined by the recursion

$$d_G(wb) = \begin{cases} 1 & \text{if } w = b = \lambda, \\ m^s d_G(w) \beta_i(\delta(w))(b) & \text{if } b \neq \lambda, \end{cases}$$

for all $w \in \Sigma_m^*$ and $b \in \Sigma_m \cup \{\lambda\}$. For $s \in [0,\infty)$, a function $d : \Sigma_m^* \to [0,\infty)$ is a *finite-state s-gale* if it is the s-gale of some finite-state gambler.

Note that in the original definition of a finite-state gambler the range of the betting function $\vec{\beta}$ is $\Delta(\{0,1\}) \cap \mathbb{Q}^2$ [6, 1]. In the following observation, we show that allowing the range of $\vec{\beta}$ to have irrational probability measures does not change the notions of finite-state dimension and strong dimension.

Observation 3.1 *Let $G = (Q, \Sigma_m, \delta, \vec{\beta}, q_0)$ be an FSG. For each $\epsilon > 0$, there exists an FSG $G = (Q, \Sigma_m, \delta, \vec{\beta}', q_0)$ with $\vec{\beta}' : Q \to \Delta(\Sigma_m) \cap \mathbb{Q}^m$ such that for all $s \in [0,\infty)$, $S^{\infty}[d_G^{(s)}] \subseteq S^{\infty}[d_{G'}^{(s+\epsilon)}]$ and $S^{\infty}_{\mathrm{str}}[d_G^{(s)}] \subseteq S^{\infty}_{\mathrm{str}}[d_{G'}^{(s+\epsilon)}]$.*

In this paper, we allow the finite-state gamblers to place irrational bets.

Definition. ([6, 1]). Let $X \subseteq \mathbf{C}_m$. The *finite-state dimension* of X is

$$\dim_{\mathrm{FS}}(X) = \inf \{s \in [0,\infty) \mid X \subseteq S^{\infty}[d] \text{ for some finite-state } s\text{-gale } d\}$$

and the *finite-state strong dimension* of X is

$$\mathrm{Dim}_{\mathrm{FS}}(X) = \inf \{s \in [0,\infty) \mid X \subseteq S^{\infty}_{\mathrm{str}}[d] \text{ for some finite-state } s\text{-gale } d\}.$$

We will use the following basic properties of the Hausdorff, packing, finite-state, strong finite-state dimensions.

Theorem 3.2. *([6, 1]). Let $X, Y, X_i \subseteq \Sigma_m^{\infty}$ for $i \in \mathbb{N}$.*

1. $0 \leq \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{FS}}(X) \leq 1$, $0 \leq \dim_{\mathrm{P}}(X) \leq \mathrm{Dim}_{\mathrm{FS}}(X) \leq 1$.
2. $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{P}}(X)$, $\dim_{\mathrm{FS}}(X) \leq \mathrm{Dim}_{\mathrm{FS}}(X)$.
3. *If $X \subseteq Y$, then the dimension of X is at most that same dimension of Y.*
4. $\dim_{\mathrm{FS}}(X \cup Y) = \max\{\dim_{\mathrm{FS}}(X), \dim_{\mathrm{FS}}(Y)\}$ *and* $\mathrm{Dim}_{\mathrm{FS}}(X \cup Y) = \max \{\mathrm{Dim}_{\mathrm{FS}}(X), \mathrm{Dim}_{\mathrm{FS}}(Y)\}$.
5. $\dim_{\mathrm{H}} \left(\bigcup_{i=0}^{\infty} X_i\right) = \sup_{i\in\mathbb{N}} \dim_{\mathrm{H}}(X_i)$, $\dim_{\mathrm{P}}(\bigcup_{i=0}^{\infty} X_i) = \sup_{i\in\mathbb{N}} \dim_{\mathrm{P}}(X_i)$.

4 Relative Frequencies of Digits

As we have mentioned in Section 1, Besicovitch in 1934 and Eggleston in 1949 proved the following two identities respectively.

Theorem 4.1. $\dim_{\mathrm{H}}(\mathrm{FREQ}^{\leq\beta}) = \mathcal{H}_2((\beta, 1-\beta))$ *[3] and* $\dim_{\mathrm{H}}(\mathrm{FREQ}_{\beta}) = \mathcal{H}_2((\beta, 1-\beta))$ *[7].*

In this section, we will calculate the finite-state dimension of some more exotic sets that contain m-adic sequences that satisfy certain conditions placed on the frequencies of digits. The proofs in this section use straightforward constructions of finite-state gamblers. Both the constructions and analysis use completely elementary techniques.

Let $\mathcal{H}_{\beta,m}(\alpha) = -(\alpha \log_m \alpha + \beta\alpha \log_m \beta\alpha + (1-\alpha-\beta\alpha)\log_m \frac{1-\alpha-\beta\alpha}{m-2})$. Let

$$\alpha^*(x) = \begin{cases} \frac{1}{m} & x < 1 \\ \frac{1}{1+x+(m-2)x^{\frac{x}{x+1}}} & \text{otherwise.} \end{cases}$$

Note that

$$\mathcal{H}_{\beta,m}(\alpha^*(\beta)) = \sup_{\alpha\in[0,\frac{1}{1+\beta}]} \mathcal{H}_{\beta,m}(\alpha) = \begin{cases} 1 & \text{if } \beta < 1, \\ \log_m(m-2+\frac{1+\beta}{\beta^{\frac{\beta}{\beta+1}}}) & \text{otherwise.} \end{cases}$$

Theorem 4.2. *Let* $\beta' \geq \beta \geq 0$. *Let*

$$X = \left\{ S \;\middle|\; \liminf_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} \geq \beta \textit{ and } \limsup_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} \geq \beta' \right\}.$$

Then $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{FS}}(X) = \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$ *and* $\dim_{\mathrm{P}}(X) = \mathrm{Dim}_{\mathrm{FS}}(X) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$.

Corollary 4.3 *(Theorem 2 [2]). Let* $\beta \geq 0$. *Let*

$$X = \left\{ S \;\middle|\; \lim_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} = \beta \right\}.$$

Let $\beta' = \max\{\beta, 1/\beta\}$. *Then*

$$\dim_{\mathrm{H}}(X) = \mathcal{H}_{\beta,m}(\alpha^*(\beta')) = \log_m \left(m - 2 + \frac{1+\beta'}{\beta^{\frac{\beta'}{\beta'+1}}} \right)$$

Note that $\dim_{\mathrm{P}}(X)$, $\dim_{\mathrm{FS}}(X)$, and $\mathrm{Dim}_{\mathrm{FS}}(X)$ all takes the value of $\dim_{\mathrm{H}}(X)$, which were not proven in [2].

Proof. We prove the case where $\beta' = \beta$. The other case is similar. Let $Y = \left\{ S \;\middle|\; \liminf_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} \geq \beta \right\}$. Let

$$Z = \left\{ S \;\middle|\; \begin{array}{l} \lim_{n\to\infty} \pi_0(S,n) = \alpha^*(\beta), \lim_{n\to\infty} \pi_1(S,n) = \beta\alpha^*(\beta), \\ \text{and } (\forall i > 1) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha^*(\beta)-\beta\alpha^*(\beta)}{m-2} \end{array} \right\}.$$

By Eggleston's theorem, $\dim_{\mathrm{H}}(Z) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$. Since $Z \subseteq X \subseteq Y$, it follows immediately from Theorem 4.2 that $\dim_{\mathrm{H}}(X) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$.

5 Saturated Sets and Maximum Entropy Principle

In Section 4, we calculated the finite-state dimensions of many sets defined using properties on asymptotic frequencies of digits. They are all saturated sets. Now we formally define saturated sets and investigate their collective properties.

Let $\Pi_n(S) = \{\vec{\pi}(S,m) \mid m \geq n\}$ for all $n \in \mathbb{N}$. Let $\bar{\Pi}_n(S) = \overline{\Pi_n(S)}$, i.e., $\bar{\Pi}_n(S)$ is the closure of $\Pi_n(S)$. Define $\Pi : \mathbf{C}_m \to \mathcal{P}(\Delta(\Sigma_m))$ such that for all $S \in \mathbf{C}_m$, $\Pi(S) = \bigcap_{n\in\mathbb{N}} \bar{\Pi}_n(S)$.

Definition. Let $X \subseteq \mathbf{C}_m$. We say that X *is saturated* if for all $S, S' \in \mathbf{C}_m$,

$$\Pi(S) = \Pi(S') \Rightarrow [S \in X \iff S' \in X].$$

When we determine an upper bound on the finite-state dimensions of a set $X \subseteq \mathbf{C}_m$, it is in general not possible to use a single probability measure as the betting strategy even when X is saturated. However, when certain conditions are true, a simple 1-state finite-state gambler may win on a huge set of sequences with different empirical digit distribution probability measures.

In the following, we formalize such a condition and reveal some relationship between betting and the Kullback-Leibler distance (relative entropy) [5]. Note that m-dimensional Kullback-Leibler distance $\mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha})$ is defined as

$$\mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha}) = \mathrm{E}_{\vec{\beta}} \log_m \frac{\vec{\beta}}{\vec{\alpha}}.$$

Definition. Let $\vec{\alpha}, \vec{\beta} \in \Delta(\Sigma_m)$. We say that $\vec{\alpha}$ ϵ-*dominates* $\vec{\beta}$, denoted as $\vec{\alpha} >>^{\epsilon} \vec{\beta}$, if $\mathcal{H}_m(\vec{\alpha}) \geq \mathcal{H}_m(\vec{\beta}) + \mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha}) - \epsilon$. We say that $\vec{\alpha}$ *dominates* $\vec{\beta}$, denoted as $\vec{\alpha} >> \vec{\beta}$, if $\vec{\alpha} >>^0 \vec{\beta}$.

Note that $\mathcal{H}_m(\vec{\beta}) + \mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha}) = \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\beta}} + \mathrm{E}_{\vec{\beta}} \log_m \frac{\vec{\beta}}{\vec{\alpha}} = \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}}$, where $\mathrm{E}_{\vec{\beta}} \log_m \frac{\vec{\beta}}{\vec{\alpha}} = \sum_{i=0}^{m-1} \beta_i \log_m \frac{\beta_i}{\alpha_i}$. It is very easy to see that the uniform probability measure dominates all probability measures.

Observation 5.1 *Let $\vec{\alpha} = (\frac{1}{m}, \ldots, \frac{1}{m})$. Let $\vec{\beta} \in \Delta(\Sigma_m)$. Then $\vec{\alpha} >> \vec{\beta}$.*

Here, we give a few interesting properties of the domination relation.

Theorem 5.2. *Let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{k-1}) \in \Delta(\Sigma_k)$. Let $\vec{\beta} = (\beta_0, \ldots, \beta_{k-1}) \in \Delta(\Sigma_k)$ be such that $\beta_j = 1$, where $j = \arg\max\{\alpha_0, \ldots, \alpha_{k-1}\}$. Then $\vec{\alpha} >> \vec{\beta}$ and $\mathcal{H}_k(\vec{\beta}) = 0$.*

Theorem 5.3. *Let $\vec{\alpha}, \vec{\beta} \in \Delta(\Sigma_k)$, $\epsilon \geq 0$, and $r \in [0,1]$. If $\vec{\alpha} >>^{\epsilon} \vec{\beta}$, then $\vec{\alpha} >>^{\epsilon} r\vec{\alpha} + (1-r)\vec{\beta}$.*

Theorem 5.4. *Let $\vec{\mu} = (\frac{1}{m}, \ldots, \frac{1}{m}) \in \Delta(\Sigma_m)$ be the uniform probability measure. Let $\vec{\beta} \in \Delta(\Sigma_m)$. Let $s \in [0,1]$. Let $\vec{\alpha} = s\vec{\mu} + (1-s)\vec{\beta}$. Then $\vec{\alpha} >> \vec{\beta}$.*

The following theorem relates the domination relation to finite-state dimensions.

Theorem 5.5. *Let $\vec{\alpha} \in \Delta(\Sigma_k)$ and $X \subseteq \Sigma_k^{\infty}$.*

1. *If $\vec{\alpha} >>^{\epsilon} \vec{\pi}(S,n)$ for infinitely many n for every $\epsilon > 0$ and every $S \in X$, then $\dim_{\mathrm{FS}}(X) \leq \mathcal{H}_k(\vec{\alpha})$.*
2. *If $\vec{\alpha} >>^{\epsilon} \vec{\pi}(S,n)$ for all but finitely many n for every $\epsilon > 0$ and every $S \in X$, then $\mathrm{Dim}_{\mathrm{FS}}(X) \leq \mathcal{H}_k(\vec{\alpha})$.*

Theorem 5.5 tells us that if we can find a single dominating probability measure for $X \subseteq \mathbf{C}_m$, then a simple 1-state FSG may be used to assess the dimension of X. However, in the following, we will see that the domination relationship is not even transitive.

Theorem 5.6. *Domination relation defined above is not transitive.*

Fix $\vec{\alpha} \in \Delta(\Sigma_m)$ with $\mathcal{H}_m(\vec{\alpha}) \neq 1$, the hyperplane H in $\mathbb{R}^m$ defined by $\mathcal{H}_m(\vec{\alpha}) = \sum_{i=0}^{m-1} x_i \log_m \frac{1}{\alpha_i}$ divides the simplex $\Delta(\Sigma_m)$ into two halves A and B with $A \cap B \subseteq H$. Suppose $(\frac{1}{m}, \ldots, \frac{1}{m}) \in B$, then $A = \{\vec{\beta} \in \Delta(\Sigma_m) \mid \vec{\alpha} >> \vec{\beta}\}$.

So it is not always possible to find a single probability measure that dominates all the empirical probability measures of sequences in $X \subseteq \mathbf{C}_m$. Nevertheless, we take advantage of the compactness of $\Delta(\Sigma_m)$ and give a general solution for finding the dimensions of $X \subseteq \mathbf{C}_m$, when X is saturated. The following theorem is our pointwise maximum entropy principle for saturated sets. It says that the dimension of a saturated set is the maximum pointwise asymptotic entropy of the empirical digit distribution measure.

Theorem 5.7. *Let $X \subseteq \mathbf{C}_m$ be saturated. Let $H = \sup_{S \in X} \liminf_{n\to\infty} \mathcal{H}_m(\vec{\pi}(S,n))$ and $P = \sup_{S \in X} \limsup_{n\to\infty} \mathcal{H}_m(\vec{\pi}(S,n))$. Then $\dim_{\mathrm{FS}}(X) = \dim_{\mathrm{H}}(X) = H$ and $\mathrm{Dim}_{\mathrm{FS}}(X) = \dim_{\mathrm{P}}(X) = P$.*

This theorem automatically gives a solution for finding an upper bounds for dimensions of arbitrary X.

Corollary 5.8 *Let $X \subseteq \mathbf{C}_m$ and let H and P be defined as in Theorem 5.7. Then $\dim_{\mathrm{FS}}(X) \leq H$ and $\mathrm{Dim}_{\mathrm{FS}}(X) \leq P$.*

In the following, we derive the dimensions of a few interesting saturated sets using Theorem 5.7. We will give more examples in the full version of this paper.

Let $H_{\alpha,m} = \log_m[\alpha^{-\alpha}(\frac{1-\alpha}{m-1})^{\alpha-1}]$.

Theorem 5.9. *Let $\underline{\alpha}, \bar{\alpha} \in [0,1]$ such that $1/m < \underline{\alpha} \leq \bar{\alpha}$ and let $M_k^{\underline{\alpha},\bar{\alpha}} = \{S \in \Sigma_m^\infty \mid \liminf_{n\to\infty} \pi_k(S,n) = \underline{\alpha}$ and $\limsup_{n\to\infty} \pi_k(S,n) = \bar{\alpha}\}$. Then $\dim_{\mathrm{H}}(M_k^{\underline{\alpha},\bar{\alpha}}) = H_{\bar{\alpha},m}$ and $\dim_{\mathrm{P}}(M_k^{\underline{\alpha},\bar{\alpha}}) = H_{\underline{\alpha},m}$.*

Proof. It is easy to check that $M_k^{\underline{\alpha},\bar{\alpha}}$ is saturated, that $H_{\bar{\alpha},m} = \inf_{\alpha \in [\underline{\alpha},\bar{\alpha}]} H_{\alpha,m}$, and that $H_{\underline{\alpha},m} = \sup_{\alpha \in [\underline{\alpha},\bar{\alpha}]} H_{\alpha,m}$. The theorem follows from Theorem 5.7.

Corollary 5.10 *Let $\underline{\alpha}, \bar{\alpha} \in [0,1]$ such that $\underline{\alpha} \leq \bar{\alpha}$ and let $M_k^{\underline{\alpha},\bar{\alpha}} = \{S \in \mathbf{C}_m \mid \liminf_{n\to\infty} \pi_k(S,n) = \underline{\alpha}$ and $\limsup_{n\to\infty} \pi_k(S,n) = \bar{\alpha}\}$. Then $\dim_{\mathrm{H}}(M_k^{\underline{\alpha},\bar{\alpha}}) = \inf_{\alpha \in [\underline{\alpha},\bar{\alpha}]} H_{\alpha,m} = \min(H_{\underline{\alpha},m}, H_{\bar{\alpha},m})$ and*

$$\dim_{\mathrm{P}}(M_k^{\underline{\alpha},\bar{\alpha}}) = \sup_{\alpha \in [\underline{\alpha},\bar{\alpha}]} H_{\alpha,m} = \begin{cases} 1 & \underline{\alpha} \leq 1/m \leq \bar{\alpha}, \\ \max(H_{\underline{\alpha},m}, H_{\bar{\alpha},m}) & \textit{otherwise.} \end{cases}$$

Proof. If $\underline{\alpha} \leq 1/m \leq \bar{\alpha}$, then for some $S \in M_k^{\underline{\alpha},\bar{\alpha}}$, $\limsup_{n\to\infty} \mathcal{H}_m(\vec{\pi}(S,n)) = 1$.

Corollary 5.11 *(Theorem 7 [2]). Let $\underline{\alpha}_k, \bar{\alpha}_k \in [0,1]$ for $k \in \Sigma_m$. Let $M_R = \bigcap_{k=0}^{m-1} M_k^{\underline{\alpha}_k,\bar{\alpha}_k}$. Then $\dim_{\mathrm{FS}}(M_R) = \dim_{\mathrm{H}}(M_R) = \min_{k=0}^{m-1} \dim_{\mathrm{H}}(M_k^{\underline{\alpha},\bar{\alpha}})$ and $\mathrm{Dim}_{\mathrm{FS}}(M_R) = \dim_{\mathrm{P}}(M_R) = \min_{k=0}^{m-1} \dim_{\mathrm{P}}(M_k^{\underline{\alpha},\bar{\alpha}})$.*

Corollary 5.12 *(Theorem 1 [2]). Let* $k \in \Sigma_m$ *and let* $M_k = \{S \in \mathbf{C}_m \mid \liminf_{n\to\infty} \pi_k(x,n) < \limsup_{n\to\infty} \pi_k(x,n)\}$. *Then* $\dim_{\mathrm{H}}(\cap_{k=0}^{m-1} M_k) = 1$.

Theorem 5.13. *Let* A *be a* $d \times m$ *matrix and* $b = (b_1, \ldots, b_d) \in \mathbb{R}^d$. *Let*

$$K^{\mathrm{i.o.}}(A,b) = \{S \in \mathbf{C}_m \mid (\exists \{k_n\} \subseteq \mathbb{N}) \lim_{n\to\infty} k_n = \infty \text{ and } \lim_{n\to\infty} A(\vec{\pi}(S,k_n))^T = b\}$$

and let $K(A,b) = \{S \in \mathbf{C}_m \mid \lim_{n\to\infty} A(\vec{\pi}(S,n))^T = b\}$. *Then* $\dim_{\mathrm{FS}}(K^{\mathrm{i.o.}}(A,b)) = \dim_{\mathrm{H}}(K^{\mathrm{i.o.}}(A,b)) = \sup_{\substack{\vec{\alpha}\in\Delta(\Sigma_m) \\ A\vec{\alpha}^T = b}} \mathcal{H}_m(\vec{\alpha})$, $\dim_{\mathrm{P}}(K^{\mathrm{i.o.}}(A,b)) = 1$, *and*

$$\dim_{\mathrm{H}}(K(A,b)) = \mathrm{Dim}_{\mathrm{FS}}(K(A,b)) = \sup_{\substack{\vec{\alpha}\in\Delta(\Sigma_m) \\ A\vec{\alpha}^T = b}} \mathcal{H}_m(\vec{\alpha}).$$

Proof. It is easy to check that $K^{\mathrm{i.o.}}(A,b)$ and $K(A,b)$ are both saturated. □

6 Conclusion

A general saturated set usually has an uncountable decomposition in which, the dimension of each element is easy to determine, while the dimension of the whole set, which is the uncountable union of all the element sets, is very difficult to determine and requires advanced techniques in multifractal analysis and ergodic theory. By using finite-state gambler and gale characterizations of dimensions, we are able to obtain very general results calculating the classical dimensions and finite-state dimensions of saturated sets using completely elementary analysis. This indicates that gale characterizations will play a more important role in dimension-theoretic analysis and that finite-state gambler is very powerful.

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A Appendix for Section 3

Proof (**Proof of Observation 3.1).** Let $\delta = \min_{q\in Q}\min_{i=0..m}(\beta_i(q) - \frac{\beta_i(q)}{2^\epsilon})$. For all $q \in Q$, let $\beta'_i(q) \in [\beta_i(q) - \delta, \beta_i(q)] \cap [0,1] \cap \mathbb{Q}$, if $i \neq 0$. Otherwise, let $\beta'_0(q) = 1 - \sum_{i=1}^{m-1}\beta'_i(q)$. Note that for all $q \in Q$ and all $i \in \{0,\dots,m-1\}$,

$$\beta'_i(q) \geq \beta_i(q) - \delta \geq 0 \tag{A.1}$$

and that β' maps states to rational bets.

For $w \in \Sigma_m^*$, let

$$\#q_i(w) = |\{n \mid 0 \leq n \leq |w|-1, \delta(w[0..n-1]) = q \text{ and } w[n] = i\}|.$$

Note that

$$|w| = \sum_{q\in Q}\sum_{i=0}^{m-1}\#q_i(w). \tag{A.2}$$

Now we have that for all $w \in \Sigma_m^*$,

$$\begin{aligned} d_{G'}^{(s+\epsilon)}(w) &= c_0 m^{(s+\epsilon)|w|}\prod_{q\in Q}\prod_{i=0}^{m-1}\beta'_i(q)^{\#q_i(w)} \\ &\geq^{(\text{A.1})} c_0 m^{(s+\epsilon)|w|}\prod_{q\in Q}\prod_{i=0}^{m-1}(\beta_i(q)-\delta)^{\#q_i(w)} \\ &=^{(\text{A.2})} c_0 m^{s|w|}\prod_{q\in Q}\prod_{i=0}^{m-1}[(\beta_i(q)-\delta)2^\epsilon]^{\#q_i(w)}. \end{aligned}$$

By the choice of δ,

$$\begin{aligned} d_{G'}^{(s+\epsilon)}(w) &\geq c_0 m^{s|w|}\prod_{q\in Q}\prod_{i=0}^{m-1}[(\beta_i(q) - (\beta_i(q) - \tfrac{\beta_i(q)}{2^\epsilon}))2^\epsilon]^{\#q_i(w)} \\ &= c_0 m^{s|w|}\prod_{q\in Q}\prod_{i=0}^{m-1}\beta_i(q)^{\#q_i(w)} \\ &= d_G^{(s)}(w). \end{aligned}$$

Thus, for all $S \in S^\infty[d_G^{(s)}]$,

$$\limsup_{n\to\infty} d_{G'}^{(s+\epsilon)}(S) \geq \limsup_{n\to\infty} d_G^{(s)}(S) = \infty,$$

and for all $S \in S^\infty_{\text{str}}[d_G^{(s)}]$,

$$\liminf_{n\to\infty} d_{G'}^{(s+\epsilon)}(S) \geq \liminf_{n\to\infty} d_G^{(s)}(S) = \infty.$$

Therefore,

$$S^{\infty}[d_G^{(s)}] \subseteq S^{\infty}[d_{G'}^{(s+\epsilon)}]$$

and

$$S^{\infty}_{\mathrm{str}}[d_G^{(s)}] \subseteq S^{\infty}_{\mathrm{str}}[d_{G'}^{(s+\epsilon)}].$$

□

B Appendix for Section 4

Proof (**Proof of Theorem 4.2**). We assume that $\beta' \geq \beta \geq 1$, since when either of these values are less than 1, the proof is essentially looking at the subset of X where their values are replaced by 1. First, we prove the lower bounds for the dimensions.

When S is clear from the context, let $\alpha_n = \pi_0(S,n)$ and $\beta_n = \pi_1(S,n)$.

Let $\alpha' = \alpha^*(\beta')$ and let $\alpha = \alpha^*(\beta)$.

For Hausdorff dimension and finite-state dimension, let

$$Y = \left\{ S \;\middle|\; \lim_{n\to\infty} \alpha_n = \alpha', \lim_{n\to\infty} \beta_n = \beta'\alpha', \text{and } (\forall i > 1) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha'-\beta'\alpha'}{m-2} \right\}.$$

By Eggleston's theorem, we have $\dim_{\mathrm{H}}(Y) = \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$. Since $\beta' \geq \beta \geq 1$ and $Y \subseteq X$,

$$\dim_{\mathrm{FS}}(X) \geq \dim_{\mathrm{H}}(X) \geq \dim_{\mathrm{H}}(Y) = \mathcal{H}_{\beta',m}(\alpha^*(\beta')).$$

For packing dimension and finite-state strong dimension, let

$$Z = \left\{ S \;\middle|\; \lim_{n\to\infty} \alpha_n = \alpha, \lim_{n\to\infty} \beta_n = \beta\alpha, \text{and } (\forall i > 1) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha-\beta\alpha}{m-2} \right\}.$$

Now we construct from Z a set $Z' \subseteq X$ by interpolating the sequences in Z.

First let $l_0 = 2$ and for every $i \in \mathbb{N}$, $l_{i+1} = 2^{l_i}$.

Define $f_0 : \Sigma_m^* \to \Sigma_m^*$ be such that $f_0(w) = w$ for all $w \in \Sigma_m^*$. Let $\rho = \frac{1}{\alpha\beta' - \alpha\beta + 1}$. For each $n > 0$, define $f_n : \Sigma_m^* \to \Sigma_m^*$ such that for every $w \in \Sigma_m^*$, $|f_n(w)| = |w|$ and for every $i < |w|$,

$$f_n(w)[i] = \begin{cases} f_{n-1}(w)[i] & i \leq l_{n-1} \\ w[i] & i \leq \lceil \rho l_n \rceil \text{ and } i > l_{n-1} \\ 1 & i > \lceil \rho l_n \rceil \text{ and } i \leq l_n \\ w[i] & i > l_n. \end{cases}$$

Define $f : \Sigma_m^* \to \Sigma_m^*$ such that for all $w \in \Sigma_m^*$

$$f(w) = f_{n(w)}(w),$$

where $n(w) = \min\{n \in \mathbb{N} \mid l_n \geq |w|\}$. Also, extend f to $f : \Sigma_m^\infty \to \Sigma_m^\infty$ such that for all $S \in \Sigma_m^\infty$,

$$f(S) = \lim_{n\to\infty} f(S[0..n-1]).$$

Let $Z' = f(Z)$.

By the construction of f and choice of ρ, it is clear that f is a dilation and for all $n \in \mathbb{N}$, $|\mathrm{Col}(f, S[0..\lceil \rho l_n \rceil - 1])| \leq \log l_n$. Thus for all $\epsilon > 0$, there are infinitely many n such that

$$|\mathrm{Col}(f, S[0..n-1])| < \epsilon n. \tag{B.1}$$

Note that by Eggleston's theorem, $\dim_{\mathrm{H}}(Z) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$. Then by Supergale Dilation Theorem [11] and (B.1), $\dim_{\mathrm{P}}(Z') \geq \mathcal{H}_{\beta,m}(\alpha^*(\beta))$.

It is easy to verify that for every $S \in Z'$,

$$\liminf_{n\to\infty} \frac{\beta_n}{\alpha_n} \geq \beta \text{ and } \limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \geq \beta'.$$

So $Z' \subseteq X$. Therefore, $\mathrm{Dim}_{\mathrm{FS}}(X) \geq \dim_{\mathrm{P}}(X) \geq \mathcal{H}_{\beta,m}(\alpha^*(\beta))$.

Now, we prove that $\mathcal{H}_{\beta',m}(\alpha^*(\beta'))$ is an upper bound for $\dim_{\mathrm{H}}(X)$ and $\dim_{\mathrm{FS}}(X)$.

If $\beta' < 1$, then $\mathcal{H}_{\beta',m}(\alpha^*(\beta')) = 1$ and the upper bound holds trivially. So assume $\beta' \geq 1$.

Let $\alpha = \alpha^*(\beta')$. Let $s > \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$. Define

$$d(wb) = \begin{cases} m^s \alpha d(w) & b = 0 \\ m^s \beta' \alpha d(w) & b = 1 \\ m^s \frac{1-\alpha-\beta'\alpha}{m-2} d(w) & b \geq 2 \end{cases}.$$

It is clear that d is a finite-state s-gale.

Let

$$B = \beta'^{\frac{\beta'}{\beta'+1}}.$$

Let

$$\epsilon = \frac{s - \mathcal{H}_{\beta',m}(\alpha^*(\beta'))}{2\log_m B}.$$

Let $S \in X$ and let $\delta > 0$ be such that $\delta \leq \min(\epsilon\beta'^2/2, 1/2)$.

Since

$$\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \geq \beta',$$

there exists an infinite set $J \subseteq \mathbb{N}$ such that for all $n \in J$

$$\frac{\beta_n}{\alpha_n} \geq \beta' - \delta.$$

By the choice of δ, for all $n \in J$

$$\frac{\alpha_n}{\beta_n} \leq \frac{1}{\beta' - \delta} = \frac{1}{\beta'} + \frac{\delta}{(\beta'-\delta)\beta'} \leq \frac{1}{\beta'} + \epsilon,$$

i.e.,

$$\alpha_n + \beta_n \leq \frac{\beta'+1}{\beta'}\beta_n + \epsilon. \tag{B.2}$$

Now, note that

$$m^s B^{1-\epsilon} = (1 + \beta' + (m-2)B)B^{\epsilon}, \tag{B.3}$$

since

$$\begin{aligned}
m^s B^{1-\epsilon} &= m^s B^{1-\frac{s-\log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\
&= B^{1+\log_B m^s - \frac{\log_m m^s - \log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\
&= B^{1+\frac{2\log_m m^s - \log_m m^s + \log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\
&= B^{1+\frac{\log_m m^s + \log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\
&= B^{1+\frac{s-\log_m(m-2+\frac{1+\beta'}{B})+2\log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\
&= B^{1+\epsilon+\log_B(m-2+\frac{1+\beta'}{B})}.
\end{aligned}$$

For all $n \in J$,

$$\begin{aligned}
d(S[0..n-1]) &= m^{sn}\alpha^{n\alpha_n}(\beta'\alpha)^{n\beta_n}\left(\frac{1-\alpha-\beta'\alpha}{m-2}\right)^{n(1-\alpha_n-\beta_n)} \\
&= \left[\frac{m^s \beta'^{\beta_n} B^{1-\alpha_n-\beta_n}}{1+\beta'+(m-2)B}\right]^n \\
&\geq^{(\text{B.2})} \left[\frac{m^s \beta'^{\beta_n} B^{1-\frac{\beta'+1}{\beta'}\beta_n-\epsilon}}{1+\beta'+(m-2)B}\right]^n \\
&= \left[\frac{m^s B^{1-\epsilon}}{1+\beta'+(m-2)B}\right]^n \\
&=^{(\text{B.3})} B^{\epsilon n}.
\end{aligned}$$

Since J is an infinite set,

$$\limsup_{n\to\infty} d(S[0..n-1]) = \infty,$$

i.e., $S \in S^{\infty}[d]$. Since $s > \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$ is arbitrary and d is finite-state s-gale, $\dim_{\text{H}}(X) \leq \dim_{\text{FS}}(X) \leq \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$.

An essentially identical argument gives us $\dim_{\text{P}}(X) \leq \text{Dim}_{\text{FS}}(X) \leq \mathcal{H}_{\beta,m}(\alpha^*(\beta))$. □

Theorem B.1. *Let* $\alpha \geq 1/m$. *Let* $X = \left\{S \;\middle|\; \lim_{n\to\infty} \pi_0(S,n) = \alpha\right\}$ *and* $Y = \left\{S \;\middle|\; \liminf_{n\to\infty} \pi_0(S,n) \geq \alpha\right\}$. *Then*

$$\dim_{\text{P}}(X) = \dim_{\text{H}}(X) = \dim_{\text{P}}(Y) = \dim_{\text{H}}(Y) = \log_m\left[\alpha^{-\alpha}\left(\frac{1-\alpha}{m-1}\right)^{\alpha-1}\right].$$

Proof (**Proof of Theorem B.1).** The results are clear for $\alpha = 1/m$, so we assume that $\alpha > 1/m$.

Let $H_{\alpha,m} = \log_m \left[\alpha^{-\alpha}\left(\frac{1-\alpha}{m-1}\right)^{\alpha-1}\right]$.

We first show that $\dim_{\mathrm{P}}(Y) \le H_{\alpha,m}$. For $s > H_{\alpha,m}$, define

$$d(wb) = \begin{cases} m^s \alpha d(w) & b = 0 \\ m^s \frac{1-\alpha}{m-1} d(w) & b \neq 0. \end{cases}$$

It is clear that d is an s-gale. Let

$$\epsilon = \frac{s - H_{\alpha,m}}{2\log_m \frac{\alpha(m-1)}{1-\alpha}}. \tag{B.4}$$

Note that $\frac{\alpha(m-1)}{1-\alpha} > 1$. Let $S \in Y$, i.e., $\liminf\limits_{n\to\infty} \pi_0(S,n) \ge \alpha$. So there exists $J \subseteq \mathbb{N}$ such that $|\mathbb{N} \setminus J| < \infty$ and for every $n \in J$,

$$\pi_0(S,n) \ge \alpha - \epsilon.$$

$$\begin{aligned} d(S[0..n-1]) &= \left[m^s \alpha^{\pi_0(S,n)} \left(\frac{1-\alpha}{m-1}\right)^{1-\pi_0(S,n)}\right]^n \\ &=^{(\mathrm{B.4})} \left[\left(\frac{\alpha(m-1)}{1-\alpha}\right)^{2\epsilon} \alpha^{-\alpha}\left(\frac{1-\alpha}{m-1}\right)^{\alpha-1} \alpha^{\pi_0(S,n)}\left(\frac{1-\alpha}{m-1}\right)^{1-\pi_0(S,n)}\right]^n \\ &= \left[\left(\frac{\alpha(m-1)}{1-\alpha}\right)^{2\epsilon} \alpha^{\pi_0(S,n)-\alpha}\left(\frac{1-\alpha}{m-1}\right)^{\alpha-\pi_0(S,n)}\right]^n \\ &= \left[\left(\frac{\alpha(m-1)}{1-\alpha}\right)^{2\epsilon} \left(\frac{\alpha(m-1)}{1-\alpha}\right)^{\pi_0(S,n)-\alpha}\right]^n \\ &= \left[\left(\frac{\alpha(m-1)}{1-\alpha}\right)^{2\epsilon+\pi_0(S,n)-\alpha}\right]^n. \end{aligned}$$

Then for every $n \in J$,

$$d(S[0..n-1]) \ge \left[\frac{\alpha(m-1)}{1-\alpha}\right]^{\epsilon n}.$$

Since $\frac{\alpha(m-1)}{1-\alpha} > 1$, $S \in S^{\infty}_{\mathrm{str}}[d]$ and $\dim_{\mathrm{H}}(Y) \le \dim_{\mathrm{P}}(Y) \le H_{\alpha,m}$. Note taht $X \subseteq Y$, so $\dim_{\mathrm{H}}(X) \le \dim_{\mathrm{P}}(X) \le H_{\alpha,m}$.

Now it suffices to show that $\dim_{\mathrm{H}}(X) \ge H_{\alpha,m}$.

Let

$$Z = \left\{ S \;\middle|\; \lim_{n\to\infty} \pi_0(S[0..n-1]) = \alpha \text{ and } (\forall i > 0) \lim_{n\to\infty} \pi_i(S[0..n-1]) = \frac{1-\alpha}{m-1} \right\}.$$

By Eggleston's theorem, $\dim_{\mathrm{H}}(Z) = H_{\alpha,m}$. Since $Z \subseteq X \subseteq Y$, $\dim_{\mathrm{H}}(Y) \ge \dim_{\mathrm{H}}(X) \ge H_{\alpha,m}$. □

Theorem B.2. *(Corollary 13 in [2]). Let Σ_m be the m-ary alphabet. Let $k < m$. Let $\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1} \in [0,1]$ be such that $\alpha = \sum_{i=0}^{k-1} \alpha_i \leq 1$. Let*

$$X = \left\{ S \;\middle|\; \lim_{n\to\infty} \pi_i(S,n) = \alpha_i, 0 \leq i \leq k \right\}.$$

Then $\dim_{\mathrm{H}}(X)$ *is*

$$\mathcal{H}_m\left(\alpha_0,\ldots,\alpha_{k-1},\tfrac{1-\alpha}{m-k},\ldots,\tfrac{1-\alpha}{m-k}\right) = \log_m\left[\alpha_0^{-\alpha_0}\cdots\alpha_{k-1}^{-\alpha_{k-1}}\left(\tfrac{1-\alpha}{m-k}\right)^{-(1-\alpha)}\right]$$

and

$$\dim_{\mathrm{FS}}(X) = \mathrm{Dim}_{\mathrm{FS}}(X) = \dim_{\mathrm{P}}(X) = \dim_{\mathrm{H}}(X).$$

Proof (**Proof of Theorem B.2).** We insist that $0^0 = 1$ and $0/0 = 1$ in the proof.

Let

$$H = \mathcal{H}_m\left(\alpha_0,\alpha_1,\ldots,\alpha_{k-1},\frac{1-\alpha}{m-k},\ldots,\frac{1-\alpha}{m-k}\right).$$

For $s > H$, define

$$d(wb) = \begin{cases} m^s d(w)\alpha_b & b < k \\ m^s d(w)\frac{1-\alpha}{m-k} & \text{otherwise.} \end{cases}$$

It is clear that d is a finite-state s-gale. Let

$$\delta = \frac{s-H}{-2\log_m(\alpha_0\cdots\alpha_{k-1}\frac{1-\alpha}{m-k})}.$$

For $S \in X$,

$$\lim_{n\to\infty} \pi_i(S,n) = \alpha_i, 0 \leq i \leq k.$$

So there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $|\pi_i(S,n) - \alpha_i| < \delta$ for all $i < k$ and that

$$\left|\alpha - \sum_{i=0}^{k-1} \pi_i(S,n)\right| < \delta$$

Then for all $n \geq n_0$,

$$
\begin{aligned}
d(S[0..n-1]) &= \left[m^s \left(\frac{1-\alpha}{m-k}\right)^{1-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)} \right]^n \\
&= \left[m^{s-H} m^H \left(\frac{1-\alpha}{m-k}\right)^{1-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)} \right]^n \\
&= \left[m^{s-H} \alpha_0^{-\alpha_0} \cdots \alpha_{k-1}^{-\alpha_{k-1}} \left(\frac{1-\alpha}{m-k}\right)^{-(1-\alpha)} \left(\frac{1-\alpha}{m-k}\right)^{1-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)} \right]^n \\
&= \left[m^{s-H} \left(\frac{1-\alpha}{m-k}\right)^{\alpha-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)-\alpha_i} \right]^n \\
&\geq \left[m^{s-H} \left(\alpha_0 \cdots \alpha_{k-1} \frac{1-\alpha}{m-k}\right)^{\delta} \right]^n = \left[m^{s-H} m^{\frac{H-s}{2}} \right]^n \\
&= m^{\frac{s-H}{2} n}.
\end{aligned}
$$

So $S \in S^{\infty}_{\text{str}}[d]$ and thus $\dim_{\text{FS}}(X) \leq \text{Dim}_{\text{FS}}(X) \leq H$.

Let

$$
Z = \left\{ S \;\middle|\; (\forall i < k) \lim_{n\to\infty} \pi_i(S,n) = \alpha_i \text{ and } (\forall i \geq k) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha}{m-k} \right\}.
$$

By Eggleston's theorem, $\dim_{\text{H}}(Z) = H$. The theorem then follows from the monotonicity of dimensions. □

C Appendix for Section 5

Proof (**Proof of Theorem 5.2).** It is easy to see that $\mathcal{H}_k(\beta) = 0$. It suffices to show that

$$
\mathcal{H}_k(\vec{\alpha}) \geq \text{E}_{\vec{\beta}} \log_k \frac{1}{\vec{\alpha}}.
$$

$$
\begin{aligned}
\text{E}_{\vec{\beta}} \log_k \frac{1}{\vec{\alpha}} &= \sum_{i=0}^{k-1} \beta_i \log_k \frac{1}{\alpha_i} = \beta_j \log_k \frac{1}{\alpha_j} \\
&= \log_k \frac{1}{\alpha_j} \leq \sum_{i=0}^{k-1} \alpha_i \log_k \frac{1}{\alpha_i} \\
&= \mathcal{H}_k(\vec{\alpha}).
\end{aligned}
$$

□

Proof (**Proof of Theorem 5.3).** Assume $\vec{\alpha} >>^{\epsilon} \vec{\beta}$, it suffices to show that

$$
\mathcal{H}_k(\vec{\alpha}) \geq \text{E}_{r\vec{\alpha}+(1-r)\vec{\beta}} \log_k \frac{1}{\vec{\alpha}} - \epsilon.
$$

$$\begin{aligned}
\mathrm{E}_{r\vec{\alpha}+(1-r)\vec{\beta}} \log_k \frac{1}{\vec{\alpha}} - \epsilon &= \sum_{i=0}^{k-1} (r\alpha_i + (1-r)\beta_i) \log_k \frac{1}{\alpha_i} - \epsilon \\
&= \sum_{i=0}^{k-1} r\alpha_i \log_k \frac{1}{\alpha_i} + \sum_{i=0}^{k-1} (1-r)\beta_i \log_k \frac{1}{\alpha_i} - \epsilon \\
&= r\mathcal{H}_k(\vec{\alpha}) + (1-r)\mathrm{E}_{\vec{\beta}} \log_k \frac{1}{\vec{\alpha}} - (1-r)\epsilon - r\epsilon \\
&\leq \mathcal{H}_k(\vec{\alpha}).
\end{aligned}$$

□

Proof (**Proof of Theorem 5.4).** Let $A = \{i \mid \mu_i \geq \beta_i\}$ and let $B = \{i \mid \mu_i < \beta_i\}$. Then $A \cap B = \varnothing$ and $A \cup B = [0..m-1]$. Note that for any $i \in A$, $\mu_i = \frac{1}{m} \geq \beta_i$ and $\log_m \frac{1}{s\mu_i + (1-s)\beta_i} \geq 1$ and for any $i \in B$, $\mu_i = \frac{1}{m} < \beta_i$ and $\sum_{i \in B} s(\mu_i - \beta_i) \log_m \frac{1}{s\mu_i+(1-s)\beta_i} < 1$. Since $\sum_{i=0}^{m-1} s(\mu_i - \beta_i) = 0$, $\sum_{i\in A} s(\mu_i - \beta_i) = -\sum_{i\in B} s(\mu_i - \beta_i)$.

$$\begin{aligned}
&\mathrm{E}_{\vec{\alpha}} \log_m \frac{1}{\vec{\alpha}} - \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}} \\
&\quad = \mathrm{E}_{s(\vec{\mu}-\vec{\beta})} \log_m \frac{1}{s\vec{\mu} + (1-s)\vec{\beta}} \\
&\quad = \sum_{i=0}^{m-1} s(\mu_i - \beta_i) \log_m \frac{1}{s\mu_i + (1-s)\beta_i} \\
&\quad = \sum_{i\in A} s(\mu_i - \beta_i) \log_m \frac{1}{s\mu_i + (1-s)\beta_i} + \sum_{i\in B} s(\mu_i - \beta_i) \log_m \frac{1}{s\mu_i + (1-s)\beta_i} \\
&\quad \geq \sum_{i\in A} s(\mu_i - \beta_i) \cdot 1 + \sum_{i\in B} s(\mu_i - \beta_i) \cdot 1 \\
&\quad \geq 0.
\end{aligned}$$

Therefore,

$$\mathrm{E}_{\vec{\alpha}} \log_m \frac{1}{\vec{\alpha}} \geq \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}},$$

i.e., $\vec{\alpha} >> \vec{\beta}$.

Proof (Proof of Theorem 5.5). Let $G = (Q, \delta, \vec{\beta}, q_0, 1)$ be an FSG such that $Q = \{q_0\}$, $\delta(q_0, b) = q_0$ for all $b \in \Sigma_k$, and $\vec{\beta}(q_0) = \vec{\alpha}$.

Let $s > \mathcal{H}_k(\vec{\alpha}) + \epsilon$. The s-gale $d_G^{(s)}$ of G is defined by the following recursion,

$$d_G^{(s)}(wb) = \begin{cases} 1 & w = b = \lambda \\ k^s d_G^{(s)}(w)\alpha_b & \text{otherwise,} \end{cases}$$

for all $w \in \Sigma_k^*$ and $b \in \Sigma_k$. Let $S \in X$. Then

$$\begin{aligned} d_G^{(s)}(S[0..n-1]) &= k^{sn} \prod_{i=0}^{k-1} \alpha_i^{n\pi_i(S,n)} \\ &= k^{sn} k^{n \sum_{i=0}^{k-1} \pi_i(S,n) \log_k \alpha_i} \\ &= \left(k^{s - \mathrm{E}_{\vec{\pi}(S,n)} \log_k \frac{1}{\vec{\alpha}}} \right)^n . \end{aligned}$$

Thus $S \in S^\infty[d_G^{(s)}]$ and $\dim_{\mathrm{FS}}(S) \leq s$, when the domination condition holds for infinitely many n. Similarly, $S \in S_{\mathrm{str}}^\infty[d_G^{(s)}]$ and $\mathrm{Dim}_{\mathrm{FS}}(S) \leq s$, when the domination condition holds for all but finitely many n. The theorem then follows, since ϵ can be arbitrarily small.

*Proof (***Proof of Theorem 5.6).** We prove this by giving a counterexample with Σ_3. This counterexample can be extended to larger alphabets very easily.

Let $\alpha = (\frac{54}{300}, \frac{54}{300}, \frac{192}{300})$, $\beta = (\frac{25}{300}, \frac{75}{300}, \frac{200}{300})$. And we have

$$\mathcal{H}(\alpha) \approx 0.8219015831,$$

and

$$\mathcal{H}(\beta) \approx 0.7344147903,$$

and

$$\mathcal{H}(\alpha) - \mathrm{E}_\beta \log_3 \frac{\beta}{\alpha} \approx 0.05003477990.$$

So $\alpha >> \beta$.

Note that fix $\alpha \in \Delta(\{0,1,2\})$, for $\gamma \in \Delta(\{0,1,2\})$, $\alpha >> \gamma$ if

$$\mathcal{H}(\alpha) \geq \mathrm{E}_\gamma \log_3 \frac{1}{\alpha},$$

i.e.,

$$\mathcal{H}(\alpha) \geq \gamma_0 \log_3 \frac{1}{\alpha_0} + \gamma_1 \log_3 \frac{1}{\alpha_1} + \gamma_2 \log_3 \frac{1}{\alpha_2}.$$

It is clear that α determines a hyperplane that separate the space of all probability measures. Since we only consider the cases where $\gamma_0 + \gamma_1 + \gamma_2 = 1$, the above inequality simplifies to

$$\mathcal{H}(\alpha) \geq \gamma_0 \log_3 \frac{1}{\alpha_0} + \gamma_1 \log_3 \frac{1}{\alpha_1} + (1 - \gamma_0 - \gamma_1) \log_3 \frac{1}{1 - \alpha_0 - \alpha_1}.$$

Let $\gamma_0 = 0$, we may solve the above inequality and obtain the boundary point for α at $\gamma_0 = 0$ is $\gamma_1 = \frac{9}{25}$. Similarly, the boundary point for β at $\gamma_0 = 0$ is approximately $\gamma_1 = 0.3965181711$.

Let $\gamma^* = (0, 0.37, 0.63)$.

$$\mathcal{H}(\alpha) - \mathrm{E}_{\gamma^*} \log_3 \frac{1}{\alpha} \approx -0.01154648767$$

and

$$\mathcal{H}(\beta) - \mathrm{E}_{\gamma^*} \log_3 \frac{1}{\beta} \approx 0.02593650702.$$

Thus β dominates γ^* but α does not dominate γ^*. □

Lemma C.1 *([12]). For every $n \geq m \geq 2$ and every partition $\vec{a} = (a_0, \dots, a_{m-1})$ of n, there are more than*

$$m^{n\mathcal{H}_m(\frac{\vec{a}}{n}) - (m+1)\log_m n}$$

strings u of length n and $\#(i,u) = a_i$ for each $i \in \Sigma_m$.

Theorem C.2. *([6]). Let d be an s-supergale, where $s \in [0, \infty)$. Then for all $w \in \Sigma_m^*$, $l \in \mathbb{N}$, and $0 < \alpha \in \mathbb{R}$, there are fewer than $\frac{m^l}{\alpha}$ strings $u \in \Sigma_m^l$ for which $d(wu) > \alpha m^{(s-1)l} d(w)$.*

Proof (**Proof of Theorem 5.7).** First we prove $\dim_{\mathrm{H}}(X) \geq H$. It suffices to show that for all $s < H$, $\dim_{\mathrm{H}}(X) \geq s$.

Let $s < H$. Let d be an arbitrary s-supergale. Let $s' = (H+s)/2$. Let $n_0 \in \mathbb{N}$ be such that $\sqrt{m} < n_0(H - s')$ and $m^{s' n_0 - (m+1)\log_m n_0} > 2^{s n_0 + 1}$.

Fix an $S \in X$ such that $\liminf\limits_{n \to \infty} \mathcal{H}_m(\vec{\pi}(S,n)) > s'$.

For each $i \geq n_0$, let $\{\vec{\beta}_{i,1}, \dots, \vec{\beta}_{i,c_i}\} \subseteq \Delta(\Sigma_m)$ be such that for each $j \in [1..c_i]$, $\vec{\beta}_{i,j} = \frac{k}{n}$ for some $k \leq n$ and $\mathcal{H}_m(\vec{\beta}_{i,j}) > s'$; for all $\vec{\beta} \in F(S)$ there exists $j \in [1..c_i]$ such that $|\vec{\beta}_{i,j} - \vec{\beta}| < 1/i$; for all $j \in [1..c_i]$, there exists $\vec{\beta} \in F(S)$ such that $|\vec{\beta}_{i,j} - \vec{\beta}| < 1/i$; for all $j \in [1..c_i - 1]$, $|\vec{\beta}_{i,j} - \vec{\beta}_{i,j+1}| < \frac{1}{i}$; for all $i \geq n_0$, $|\vec{\beta}_{i+1,0} - \vec{\beta}_{i,c_i}| < \frac{1}{i+1}$.

Now, we first construct a sequence $S' \in \Sigma_m^\infty$ by building its prefixes inductively.

Let w_0 be such that $|w_0| = 2^{n_0}$. Note that the choice of w_0 does not affect the argument, since w_0 does not change the asymptotic behavior of the sequence. Without loss of generality, assume $\vec{\pi}(w_0, |w_0|) = \beta_{n_0,1}$.

For all $n > 0$, assume w_{n-1} is already constructed. Let $w_{n,0} = w_{n-1}$. We construct inductively $w_{n,1}, \dots, w_{n,c_n}$ and then let $w_n = w_{n,c_n}$.

For $j > 0$, assume $w_{n,j-1}$ is already constructed.

Let $l = n_0 + n - 1$.

For each l, j, let

$$B_{l,j} = \left\{ u \in \Sigma_m^l \;\middle|\; \vec{\pi}(u,l) = \vec{\beta}_{l,j} \right\}.$$

For each $l \geq n_0$ and $w \in \Sigma_m^*$, let

$$W_{l,w} = \left\{ u \in \Sigma_m^l \;\middle|\; d(wu) \leq \frac{1}{m} d(w) \right\}.$$

Since d is an s-supergale, by Theorem C.2, for all $w \in \Sigma_m^*$, there are fewer than m^{sl+1} strings $u \in \Sigma_m^l$ for which $d(wu) > \frac{1}{m} d(w)$. By the choice of n_0, $\vec{\beta}_{l,j}$, and Lemma C.1,

$$|B_{l,j}| > m^{sl+1},$$

i.e., $W_{l,w} \cap B_{l,j} \neq \varnothing$.

Let $u_1 \in W_{l,w} \cap B_{l,j}$. For all $i \in [2..2^{|w_{n,j-1}|}]$, let $u_i \in W_{l,wu_1\ldots u_{i-1}} \cap B_{l,j}$.

Let $w_{n,j} = w_{n,j-1}u_1 \ldots u_{2^{|w_{n,j-1}|}}$.

Let $S' = \lim_{n\to\infty} w_n$.

Note that when w_n is being constructed, $l \leq \lfloor \log_m |w_{n,j-1}| \rfloor$. It is then easy to verify that $S' \notin S^\infty[d]$.

Now we verify that $\Pi(S) = \Pi(S')$. Then $S' \in X$, since X is defined by asymptotic frequency.

Let $\vec{\beta} \in \Pi(S)$ be arbitrary. For each $l = n_0 + n - 1$, there exists some j_l such that $|\vec{\beta} - \vec{\beta}_{l,j_l}| < \frac{1}{l}$. Then by the construction,

$$|\vec{\pi}(w_{l,j_l}, |w_{l,j_l}|) - \vec{\beta}_{l,j_l}| < \sqrt{m}\frac{2}{|w_{l,j_l}|} < \frac{1}{l}.$$

So it is clear that

$$|\vec{\pi}(w_{l,j_l}, |w_{l,j_l}|) - \vec{\beta}| < \frac{2\sqrt{m}}{l}.$$

Thus

$$\lim_{l\to\infty} \vec{\pi}(w_{l,j_l}, |w_{l,j_l}|) = \vec{\beta}.$$

Since $w_{l,j_l} \sqsubseteq S'$ for all $l = n_0 + n - 1$. So we have for all $n \in \mathbb{N}$, $\vec{\beta} \in \bar{\Pi}_n(S')$, hence $\vec{\beta} \in \Pi(S')$. Therefore $\Pi(S) \subseteq \Pi(S')$.

Now, let $\vec{\beta} \notin \Pi(S)$. Since $\Pi(S)$ is closed, there exists $\delta > 0$ such that for all $\vec{\beta}' \in \Pi(S)$, $|\vec{\beta} - \vec{\beta}'| > \delta$.

Let n_1 be such that $l_1 = n_0 + n_1 - 1 > \frac{8m}{\delta}$. By construction, for all $l \geq l_1$, all $j \in [1..c_l]$, and all $|w_{l,j-1}| \leq k \leq |w_{l,j}|$,

$$|\vec{\pi}(w_{l,j}, |w_{l,j}|) - \vec{\pi}(w_{l,j}, k)| < \frac{2\sqrt{m}}{l}.$$

Also, for all $l \geq l_1$ and all $j \in [1..c_l]$, there exists $\vec{\beta}' \in \Pi(S)$ such that

$$|\vec{\pi}(w_{l,j}, |w_{l,j}|) - \vec{\beta}'| < \frac{2\sqrt{m}}{l}.$$

Thus for all $k > |w_{l_1,1}|$, there exists $\vec{\beta}' \in \Pi(S)$ such that

$$|\vec{\pi}(S, k) - \vec{\beta}'| < \frac{4m}{l}.$$

Therefore, for all $k > |w_{l_1,1}|$

$$|\vec{\pi}(S, k) - \vec{\beta}'| < \frac{4m}{l_1} < \frac{\delta}{2}.$$

Thus for all sufficiently large k,

$$|\vec{\pi}(S, k) - \vec{\beta}| > \frac{\delta}{2}.$$

So there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, $\vec{\beta} \notin \bar{\Pi}_n$, i.e., $\vec{\beta} \notin \Pi(S')$.

Now we have that $S' \in X$. Since $S' \notin S^{\infty}[d]$, $s < H$ is arbitrary, and d is an arbitrary s-supergale,

$$\dim_{\mathrm{H}}(X) \geq H.$$

By a similar construction, we may prove that

$$\dim_{\mathrm{P}}(X) \geq P.$$

In the following, we prove the finite-state dimension upper bounds. Given $\vec{\alpha} \in \Delta(\Sigma_m)$, define $B(\vec{\alpha}, r)$ as

$$B(\vec{\alpha}, r) = \Delta(\Sigma_m) \cap \left\{ \vec{\beta} \in \mathbb{R}^m \;\middle|\; (\forall i)[\beta_i < \alpha_i m^r \text{ and } \beta_i > \alpha_i m^{-r}] \right\}.$$

Let

$$F(X) = \{ \vec{\alpha} \in \Delta(\Sigma_m) \mid \mathcal{H}(\vec{\alpha}) = H \}.$$

Let $\epsilon > 0$. Let

$$\mathcal{C} = \left\{ B(\vec{\alpha}, \tfrac{\epsilon}{2}) \mid \vec{\alpha} \in F(X) \right\}.$$

It is clear that $\mathcal{C}$ is an open cover of $F(X)$. Since $F(X)$ is compact, there exists $C \subseteq \Delta(\Sigma_m)$ such that $|C| < \infty$ and

$$F(X) \subseteq \bigcup_{\vec{\alpha} \in C} B(\vec{\alpha}, \tfrac{\epsilon}{2}).$$

Let $S \in X$. Then $\liminf\limits_{n \to \infty} \mathcal{H}_m(\vec{\pi}(S, n)) \leq H$. By Theorem 5.4, there exists $\vec{\alpha}^* \in F(X)$ such that $\vec{\alpha}^* >>^{\epsilon} \vec{\pi}(S, n)$ for infinitely many $n \in \mathbb{N}$. Then by the construction of C, there exists $\vec{\alpha} \in C$ such that $\vec{\alpha}^* \in B(\vec{\alpha}, \frac{\epsilon}{2})$. Now, we have that for infinitely many $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{H}_m(\vec{\alpha}) = \mathcal{H}_m(\vec{\alpha}^*) &\geq \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{1}{\vec{\alpha}^*} - \frac{\epsilon}{2} \\
&= \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{1}{\vec{\alpha}} + \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{\vec{\alpha}}{\vec{\alpha}^*} - \frac{\epsilon}{2}.
\end{aligned}$$

By the definition of $B(\alpha, \frac{\epsilon}{2})$,

$$\mathcal{H}_m(\vec{\alpha}) \geq \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{1}{\vec{\alpha}} - \epsilon,$$

i.e., $\vec{\alpha} >>^{\epsilon} \vec{\pi}(S, n)$ for infinitely many $n \in \mathbb{N}$. Since $S \in X$ is arbitrary, we may partition X as $X = \bigcup_{\vec{\alpha} \in C} X_{\vec{\alpha}}$ such that for every $\vec{\alpha} \in C$,

$$X_{\vec{\alpha}} = \{ S \in X \mid \vec{\alpha} >>^{\epsilon} \vec{\pi}(S, n) \text{ for infinitely many } n \in \mathbb{N} \}.$$

Since $\epsilon > 0$ is arbitrary, thus by Theorem 5.5, $\dim_{\mathrm{FS}}(X_{\vec{\alpha}}) \leq \mathcal{H}_m(\vec{\alpha}) = H$ for every $\vec{\alpha} \in C$. Since $|C| < \infty$, by Theorem 3.2, $\dim_{\mathrm{FS}}(X) \leq H$. Similarly, $\mathrm{Dim}_{\mathrm{FS}}(X) \leq P$. □