# The Density of Weakly Complete Problems under Adaptive Reductions*

Jack H. Lutz
Department of Computer Science
Iowa State University
Ames, IA 50011 U.S.A.
lutz@cs.iastate.edu

Yong Zhao
LANshark Systems
784 Morrison Road
Columbus, OH 43230 U.S.A.
yzhao@lanshark.com

**Abstract**

Given a real number $\alpha < 1$, every language that is weakly $\leq^{\mathrm{P}}_{n^{\alpha/2}-\mathrm{T}}$-hard for E or weakly $\leq^{\mathrm{P}}_{n^{\alpha}-\mathrm{T}}$-hard for $\mathrm{E}_2$ is shown to be exponentially dense. This simultaneously strengthens results of Lutz and Mayordomo(1994) and Fu(1995).

## 1 Introduction

In the mid-1970's, Meyer[15] proved that every $\leq^{\mathrm{P}}_{\mathrm{m}}$-complete language for exponential time—in fact, every $\leq^{\mathrm{P}}_{\mathrm{m}}$-hard language for exponential time—is dense. That is,

$$\mathrm{E} \not\subseteq \mathrm{P}_{\mathrm{m}}(\mathrm{DENSE}^c), \tag{1}$$

where $\mathrm{E} = \mathrm{DTIME}(2^{\mathrm{linear}})$, DENSE is the class of all dense languages, $\mathrm{DENSE}^c$ is the complement of DENSE, and $\mathrm{P}_{\mathrm{m}}(\mathrm{DENSE}^c)$ is the class of all languages that are $\leq^{\mathrm{P}}_{\mathrm{m}}$-reducible to non-dense languages. (A language $A \in \{0,1\}^*$ is *dense* if there is a real number $\epsilon > 0$ such that $|A_{\leq n}| > 2^{n^\epsilon}$ for all sufficiently large $n$, where $A_{\leq n} = A \cap \{0,1\}^{\leq n}$.) Since that time, a major objective of computational complexity theory has been to extend Meyer's result from $\leq^{\mathrm{P}}_{\mathrm{m}}$-reductions to $\leq^{\mathrm{P}}_{\mathrm{T}}$-reductions, i.e., to prove that every $\leq^{\mathrm{P}}_{\mathrm{T}}$-hard language for E is dense. That is, the objective is to prove that

$$\mathrm{E} \not\subseteq \mathrm{P}_{\mathrm{T}}(\mathrm{DENSE}^c), \tag{2}$$

where $\mathrm{P}_{\mathrm{T}}(\mathrm{DENSE}^c)$ is the class of all languages that are $\leq^{\mathrm{P}}_{\mathrm{T}}$-reducible to non-dense languages. The importance of this objective derives largely from the fact (noted by Meyer[15]) that the class $\mathrm{P}_{\mathrm{T}}(\mathrm{DENSE}^c)$ contains all languages that have subexponential circuit-size complexity. (A language $A \subseteq \{0,1\}^*$ has *subexponential circuit-size complexity* if, for every real number $\epsilon > 0$, for every sufficiently large $n$, there is an $n$-input, 1-output Boolean

*This research was supported in part by National Science Foundation Grant CCR-9157382, with matching funds from Rockwell International, Microware Systems Corporation, and Amoco Foundation.

circuit that decides that the set $A_{=n} = A \cap \{0,1\}^n$ and has fewer than $2^{n^\epsilon}$ gates. Otherwise, we say that $A$ has *exponential circuit-size complexity.*) Thus a proof of (2) would tell us that E contains languages with exponential circuit-size complexity, thereby answering a major open question concerning the relationship between (uniform) time complexity and (nonuniform) circuit-size complexity. Of course (2) also implies the more modest, but more famous conjecture, that

$$\mathrm{E} \not\subseteq \mathrm{P_T}(\mathrm{SPARSE}), \tag{3}$$

where SPARSE is the class of all sparse languages. (A language $A \subseteq \{0,1\}^*$ is *sparse* if there is a polynomial $q(n)$ such that $|A_{\leq n}| \leq q(n)$ for all $n \in \mathbb{N}$.) As noted by Meyer[15], the class $\mathrm{P_T}(\mathrm{SPARSE})$ consists precisely of all languages that have polynomial circuit-size complexity, so (3) asserts that E contains languages that do not have polynomial circuit-size complexity.

Knowing (1) and wanting to prove (2), the natural strategy has been to prove results of the form

$$\mathrm{E} \not\subseteq \mathrm{P}_r(\mathrm{DENSE}^c)$$

for successively larger classes $\mathrm{P}_r(\mathrm{DENSE}^c)$ in the range

$$\mathrm{P_m}(\mathrm{DENSE}^c) \subseteq \mathrm{P}_r(\mathrm{DENSE}^c) \subseteq \mathrm{P_T}(\mathrm{DENSE}^c).$$

The first major step beyond (1) in this program was the proof by Watanabe[17] that

$$\mathrm{E} \not\subseteq \mathrm{P}_{O(\log n)-\mathrm{tt}}(\mathrm{DENSE}^c), \tag{4}$$

i.e., that every language that is $\leq^{\mathrm{P}}_{O(\log n)-\mathrm{tt}}$-hard for E is dense. The next big step was the proof by Lutz and Mayordomo[10] that, for every real number $\alpha < 1$,

$$\mathrm{E} \not\subseteq \mathrm{P}_{n^\alpha-\mathrm{tt}}(\mathrm{DENSE}^c). \tag{5}$$

This improved Watanabe's result from $O(\log n)$ truth-table (i.e., nonadaptive) queries to $n^\alpha$ such queries for $\alpha$ arbitrarily close to 1 (e.g., to $n^{0.99}$ truth-table queries). Moreover, Lutz and Mayordomo[10] proved (5) by first proving the stronger result that for all $\alpha < 1$,

$$\mu_{\mathrm{p}}(\mathrm{P}_{n^\alpha-\mathrm{tt}}(\mathrm{DENSE}^c)) = 0, \tag{6}$$

which implies that every language that is weakly $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard for E or for $\mathrm{E_2} = \mathrm{DTIME}(2^{\mathrm{poly}})$ is dense. (A language $A$ is *weakly* $\leq^{\mathrm{P}}_r$*-hard* for a complexity class $\mathcal{C}$ if $\mu(\mathrm{P}_r(A) \mid \mathcal{C}) \neq 0$, i.e., if $\mathrm{P}_r(A) \cap \mathcal{C}$ is a nonnegligible subset of $\mathcal{C}$ in the sense of the resource-bounded measure developed by Lutz[9]. A language $A$ is *weakly* $\leq^{\mathrm{P}}_r$*-complete* for $\mathcal{C}$ if $A \in \mathcal{C}$ and $A$ is weakly $\leq^{\mathrm{P}}_r$-hard for $\mathcal{C}$. See [12] or [2] for a survey of resource-bounded measure and weak completeness.) The set of weakly $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard languages for E is now known to have p-measure 1 [3], hence measure 1 in the class $\mathbf{C}$ of all languages, while the set of all $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard languages for E has measure 0 unless $\mathrm{E} \subseteq \mathrm{BPP}$ [4, 1]. Thus, if $\mathrm{E} \not\subseteq \mathrm{BPP}$ (which is generally conjectured to be true), almost every language is weakly $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard, but not $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard, for E, so the result of Lutz and Mayordomo [10] is much more general than the fact that every $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard language for E is dense.

A word on the relationship between hardness notions for E and $E_2$ is in order here. It is well known that a language is $\leq^{\mathrm{P}}_{\mathrm{m}}$-hard for E if and only if it is $\leq^{\mathrm{P}}_{\mathrm{m}}$-hard for $E_2$; this is because $E_2 = \mathrm{P_m}(E)$. The same equivalence holds for $\leq^{\mathrm{P}}_{\mathrm{T}}$-hardness. It is also clear that every language that is $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard for $E_2$ is $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard for E. However, it is not generally the case that $\mathrm{P_m}(\mathrm{P}_{n^\alpha-\mathrm{tt}}(A)) = \mathrm{P}_{n^\alpha-\mathrm{tt}}(A)$, so it may well be the case that a language can be $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard for E, but not for $E_2$. These same remarks apply to $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{T}}$-hardness.

The relationship between weak hardness notions for E and $E_2$ is somewhat different. Juedes and Lutz [8] have shown that weak $\leq^{\mathrm{P}}_{\mathrm{m}}$-hardness for E implies weak $\leq^{\mathrm{P}}_{\mathrm{m}}$-hardness for $E_2$, and their proof of this fact also works for weak $\leq^{\mathrm{P}}_{\mathrm{T}}$-hardness. However, Juedes and Lutz [8] also showed that weak $\leq^{\mathrm{P}}_{\mathrm{m}}$-hardness for $E_2$ does not generally imply weak $\leq^{\mathrm{P}}_{\mathrm{m}}$-hardness for E, and it is reasonable to conjecture (but has not been proven) that the same holds for weak $\leq^{\mathrm{P}}_{\mathrm{T}}$-hardness. We further conjecture that the notions of weak $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hardness for E and weak $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hardness $E_2$ are incomparable, and similarly for weak $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{T}}$-hardness. In any case, (6) implies that, for every $\alpha < 1$, every language that is weakly $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{tt}}$-hard for either E or $E_2$ is dense.

Shortly after, but independently of [10], Fu[7] used very different techniques to prove that, for every $\alpha < 1$,

$$\mathrm{E} \not\subseteq \mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c) \tag{7}$$

and

$$\mathrm{E}_2 \not\subseteq \mathrm{P}_{n^{\alpha}-\mathrm{T}}(\mathrm{DENSE}^c). \tag{8}$$

That is, every language that is $\leq^{\mathrm{P}}_{n^{\alpha/2}-\mathrm{T}}$-hard for E or $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{T}}$-hard for $E_2$ is dense. These results do not have the measure-theoretic strength of (6), but they are a major improvement over previous results on the densities of hard languages in that they hold for Turing reductions, which have *adaptive* queries.

In the present paper, we prove results which simultaneously strengthen results of Lutz and Mayordomo[10] and the results of Fu[7]. Specifically, we prove that, for every $\alpha < 1$,

$$\mu_{\mathrm{p}}(\mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c)) = 0 \tag{9}$$

and

$$\mu_{\mathrm{p}_2}(\mathrm{P}_{n^{\alpha}-\mathrm{T}}(\mathrm{DENSE}^c)) = 0. \tag{10}$$

These results imply that every language that is weakly $\leq^{\mathrm{P}}_{n^{\alpha/2}-\mathrm{T}}$-hard for E or weakly $\leq^{\mathrm{P}}_{n^{\alpha/2}-\mathrm{T}}$-hard for $E_2$ is dense. The proof of (9) and (10) is not a simple extension of the proof in [10] or the proof in [7], but rather combines ideas from both [10] and [7] with the martingale dilation technique introduced by Ambos-Spies, Terwijn, and Zheng [3].

Our results also show that the strong hypotheses $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$ and $\mu_{\mathrm{p}_2}(\mathrm{NP}) \neq 0$ (surveyed in [12] and [2]) have consequences for the densities of adaptively hard languages for NP. Mahaney [13] proved that

$$\mathrm{P} \neq \mathrm{NP} \Rightarrow \mathrm{NP} \not\subseteq \mathrm{P_m}(\mathrm{SPARSE}), \tag{11}$$

and Ogiwara and Watanabe [16] improved this to

$$\mathrm{P} \neq \mathrm{NP} \Rightarrow \mathrm{NP} \not\subseteq \mathrm{P}_{\mathrm{btt}}(\mathrm{SPARSE}). \tag{12}$$

That is, if $\mathrm{P} \neq \mathrm{NP}$, then no sparse language can be $\leq^{\mathrm{P}}_{\mathrm{btt}}$-hard for NP. Lutz and Mayordomo [10] used (6) to obtain a stronger conclusion from a stronger hypothesis, namely, for all $\alpha < 1$,

$$\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0 \Rightarrow \mathrm{NP} \not\subseteq \mathrm{P}_{n^{\alpha}-\mathrm{tt}}(\mathrm{DENSE}^c). \tag{13}$$

By (9) and (10), we now have, for all $\alpha < 1$,

$$\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0 \Rightarrow \mathrm{NP} \not\subseteq \mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c) \tag{14}$$

and

$$\mu_{\mathrm{p}_2}(\mathrm{NP}) \neq 0 \Rightarrow \mathrm{NP} \not\subseteq \mathrm{P}_{n^{\alpha}-\mathrm{T}}(\mathrm{DENSE}^c). \tag{15}$$

Thus, if $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, then every language that is $\leq^{\mathrm{P}}_{n^{0.49}-\mathrm{T}}$-hard for NP is dense. If $\mu_{\mathrm{p}_2}(\mathrm{NP}) \neq 0$, then every language that is $\leq^{\mathrm{P}}_{n^{0.99}-\mathrm{T}}$-hard for NP is dense.

## 2 Preliminaries

The *Boolean value* of a condition, $\psi$ is

$$[\![\psi]\!] = \begin{cases} 1 & \text{if } \psi \\ 0 & \text{if not } \psi. \end{cases}$$

The *standard enumeration* of $\{0,1\}^*$ is $s_0 = \lambda, s_1 = 0, s_2 = 1, s_3 = 00, \ldots$ This enumeration induces a total ordering of $\{0,1\}^*$ which we denote by $<$.

All *languages* here are subsets of $\{0,1\}^*$. The *Cantor space* is the set $\mathbf{C}$ of all languages. We identify each language $A \in \mathbf{C}$ with its characteristic sequence, which is the infinite binary sequence

$$[\![s_0 \in A]\!][\![s_1 \in A]\!][\![s_2 \in A]\!] \cdots,$$

where $s_0 = \lambda, s_1 = 0, s_2 = 1, s_3 = 00, \ldots$ is the standard enumeration of $\{0,1\}^*$. For $w \in \{0,1\}^*$ and $A \in \mathbf{C}$, we write $w \sqsubseteq A$ to indicate that $w$ is a prefix of (the characteristic sequence of) $A$. The *symmetric difference* of the two languages $A$ and $B$ is $A \triangle B = (A - B) \cup (B - A)$.

The *cylinder generated* by a string $w \in \{0,1\}^*$ is the set

$$\mathbf{C}_w = \{A \in \mathbf{C} | w \sqsubseteq A\}.$$

Note that $\mathbf{C}_\lambda = \mathbf{C}$.

In this paper, a set $X \subseteq \mathbf{C}$ that appears in a probability $\Pr(X)$ or a conditional probability $\Pr(X|\mathbf{C}_w)$ is regarded as an event in the sample space $\mathbf{C}$ with the uniform probability measure. Thus, for example, $\Pr(X)$ is the probability that $A \in X$ when the language $A \subseteq \{0,1\}^*$ is chosen probabilistically by using an independent toss of a fair coin to decide membership of each string in $A$. In particular, $\Pr(\mathbf{C}_w) = 2^{-|w|}$. The *complement* of a set $X \subseteq \mathbf{C}$ is the set $X^c = \mathbf{C} - X$.

Let $d \in \mathbb{N}$ and $t : \mathbb{N} \to \mathbb{N}$. A function $f : \mathbb{N}^d \times \{0,1\}^* \to \mathbb{Q}$ is *exactly* $t(n)$*-time-computable* if there is an algorithm that, on input $(k_1, \ldots, k_d, w) \in \mathbb{N}^d \times \{0,1\}^*$, runs for at most $O(t(k_1 + \cdots + k_d + |w|))$ steps and outputs an ordered pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(k_1, \ldots, k_d, w) = \frac{a}{b}$. A function $f : \mathbb{N}^d \times \{0,1\}^* \to \mathbb{R}$ is $t(n)$*-time-computable* if there is an exactly $t(n)$-time-computable function $\widehat{f} : \mathbb{N}^{d+1} \times \{0,1\}^* \to \mathbb{Q}$ such that, for all $r, k_1, \ldots, k_d \in \mathbb{N}$ and $w \in \{0,1\}^*$,

$$|\widehat{f}(r, k_1, \ldots, k_d, w) - f(k_1, \ldots, k_d, w)| \leq 2^{-r}.$$

We briefly review those aspects of martingales and resource-bounded measure that are needed for our main theorem. The reader is referred to [2], [9], [12], or [14] for more thorough discussion.

A *martingale* is a function $d : \{0,1\}^* \to [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$d(w) = \frac{d(w0) + d(w1)}{2}.$$

If $t : \mathbb{N} \to \mathbb{N}$, then a $t(n)$*-martingale* is a martingale that is $t(n)$-time-computable, and an *exact* $t(n)$*-martingale* is a (rational-valued) martingale that is exactly $t(n)$-time-computable. A martingale $d$ *succeeds* on a language $A \in \mathbf{C}$ if, for every $c \in \mathbb{N}$, there exists $w \sqsubseteq A$ such that $d(w) > c$. The *success set* of a martingale $d$ is the set

$$S^\infty[d] = \{A \in \mathbf{C} | d \text{ succeeds on } A\}.$$

The *unitary success set* of $d$ is

$$S^1[d] = \bigcup_{\substack{w \in \{0,1\}^* \\ d(w) \geq 1}} \mathbf{C}_w.$$

The following result was proven by Juedes and Lutz [8] and independently by Mayordomo [14].

**Lemma 2.1** (Exact Computation Lemma) *Let* $t : \mathbb{N} \to \mathbb{N}$ *be nondecreasing with* $t(n) \geq n^2$. *Then, for every* $t(n)$*-martingale* $d$*, there is an exact* $n \cdot t(2n+2)$*-martingale* $\widetilde{d}$ *such that* $S^\infty[d] \subseteq S^\infty[\widetilde{d}]$.

A sequence

$$\sum_{k=0}^{\infty} a_{j,k} \qquad (j = 0, 1, 2, \ldots)$$

of series of terms $a_{j,k} \in [0, \infty)$ is *uniformly* p*-convergent* if there is a polynomial $m : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $j, r \in \mathbb{N}$, $\sum_{k=m_j(r)}^{\infty} a_{j,k} \leq 2^{-r}$, where we write $m_j(r) = m(j, r)$. The following sufficient condition for uniform p-convergence is easily verified by routine calculus.

**Lemma 2.2** *Let $a_{j,k} \in [0,\infty)$ for all $j,k \in \mathbb{N}$. If there exist a real number $\epsilon > 0$ and a polynomial $g : \mathbb{N} \to \mathbb{N}$ such that $a_{j,k} \le e^{-k^{\epsilon}}$ for all $j,k \in \mathbb{N}$ with $k \ge g(j)$, then the series $\sum_{k=0}^{\infty} a_{j,k}$ $(j = 0,1,2,\dots)$ are uniformly* p*-convergent.*

A uniform, resource-bounded generalization of the classical first Borel-Cantelli lemma was proved by Lutz [9]. Here we use the following precise variant of this result.

**Theorem 2.3** *Let $\alpha, \widetilde{\alpha} \in \mathbb{R}$ with $1 \le \alpha \le \widetilde{\alpha}$, and let*

$$d : \mathbb{N} \times \mathbb{N} \times \{0,1\}^* \to \mathbb{Q} \cap [0,\infty)$$

*be an exactly $2^{(\log n)^{\alpha}}$-time-computable function with the following two properties.*

(i) *For each $j,k \in \mathbb{N}$, the function $d_{j,k}$ defined by $d_{j,k}(w) = d(j,k,w)$ is a martingale.*

(ii) *The series $\sum_{k=0}^{\infty} d_{j,k}$ $(j = 0,1,2,\dots)$ are uniformly* p*-convergent.*

*Then there is an exact $2^{(\log n)^{\widetilde{\alpha}}}$-martingale $\widetilde{\alpha}$ such that*

$$\bigcup_{j=0}^{\infty} \bigcap_{t=0}^{\infty} \bigcup_{k=t}^{\infty} S^1[d_{j,k}] \subseteq S^{\infty}[\widetilde{d}].$$

**Proof** (sketch). Assume the hypothesis, and fix $\alpha' \in \mathbb{Q}$ such that $\alpha < \alpha' < \widetilde{\alpha}$. Since $n \cdot 2^{(\log(2n+2))^{\alpha'}} = o(2^{(\log n)^{\widetilde{\alpha}}})$, it suffices by Lemma 2.1 to show that there is a $2^{(\log n)^{\alpha'}}$-martingale $d'$ such that

$$\bigcup_{j=0}^{\infty} \bigcap_{t=0}^{\infty} \bigcup_{k=t}^{\infty} S^1[d_{j,k}] \subseteq S^{\infty}[d']. \tag{16}$$

Fix a polynomial $m : \mathbb{N}^2 \to \mathbb{N}$ testifying that the series $\sum_{k=0}^{\infty} d_{j,k}$ $(j = 0,1,2,\dots)$ are uniformly p-convergent, and define

$$d'(w) = \sum_{j=0}^{\infty} \sum_{t=0}^{\infty} \sum_{k=m_j(2t)}^{\infty} 2^{t-j} d_{j,k}(w)$$

for all $w \in \{0,1\}^*$. Then, for each $w \in \{0,1\}^*$,

$$\begin{aligned} d'(w) &\le \sum_{j=0}^{\infty} \sum_{t=0}^{\infty} \sum_{k=m_j(2t)}^{\infty} 2^{t-j+|w|} d_{j,k}(\lambda) \\ &\le 2^{|w|} \sum_{j=0}^{\infty} 2^{-j} \sum_{t=0}^{\infty} 2^t \cdot 2^{-2t} \\ &= 2^{|w|+2}, \end{aligned}$$

so $d' : \{0,1\}^* \to [0,\infty)$. It is clear by linearity that $d'$ is a martingale. To see that (16) holds, assume that $A \in \bigcup_{j=0}^{\infty} \bigcap_{t=0}^{\infty} \bigcup_{k=t}^{\infty} S^1[d_{j,k}]$, and let $c \in \mathbb{N}$ be arbitrary. Then there exist $j \in \mathbb{N}$ and $k \geq m_j(2j+2c)$ such that $A \in S^1[d_{j,k}]$. Fix $w \sqsubseteq A$ such that $d_{j,k}(w) \geq 1$. Then $d'(w) \geq 2^{c+j-j} d_{j,k}(w) \geq 2^c$. Since $c$ is arbitrary here, it follows that $A \in S^\infty[d']$, confirming (16).

To see that $d'$ is $2^{(\log n)^{\alpha'}}$-time-computable, define $d_A, d_B, d_C : \mathbb{N} \times \{0,1\}^* \to [0,\infty)$ as follows, using the abbreviation $s = r + |w| + 2$.

$$\begin{aligned}
d_A(r,w) &= \sum_{j=0}^{s} \sum_{t=0}^{\infty} \sum_{k=m_j(2t)}^{\infty} 2^{t-j} d_{j,k}(w) \\
d_B(r,w) &= \sum_{j=0}^{s} \sum_{t=0}^{2s} \sum_{k=m_j(2t)}^{\infty} 2^{t-j} d_{j,k}(w) \\
d_C(r,w) &= \sum_{j=0}^{s} \sum_{t=0}^{2s} \sum_{k=m_j(2t)}^{m_j(2s^2+4s+t)} 2^{t-j} d_{j,k}(w)
\end{aligned} \tag{17}$$

For all $r \in \mathbb{N}$ and $w \in \{0,1\}^*$, it is clear that

$$d_C(r,w) \leq d_B(r,w) \leq d_A(r,w) \leq d'(w),$$

and it is routine to verify the inequalities

$$\begin{aligned}
d'(w) - d_A(r,w) &\leq 2^{-(r+1)}, \\
d_A(r,w) - d_B(r,w) &\leq 2^{-(r+2)}, \\
d_B(r,w) - d_C(r,w) &\leq 2^{-(r+2)},
\end{aligned}$$

whence we have

$$d'(w) - 2^{-r} \leq d_C(r,w) \leq d'(w) \tag{18}$$

for all $r \in \mathbb{N}$ and $w \in \{0,1\}^*$. Using formula (17), the time required to compute $d_C(r,w)$ exactly is no greater than

$$O((s+1)(2s+1)m(s, 2s^2+4s+2s)2^{(\log n)^\alpha}) = O(q(n) \cdot 2^{(\log n)^\alpha}),$$

where $n = r + |w|$ and $q$ is a polynomial. Since $q(n) \cdot 2^{(\log n)^\alpha} = o(2^{(\log n)^{\alpha'}})$, it follows that $d_C(r,w)$ is exactly $2^{(\log n)^{\alpha'}}$-time-computable. By (18), then, $d'$ is a $2^{(\log n)^{\alpha'}}$-martingale. □

The proof of our main theorem uses the techniques of weak stochasticity and martingale dilation, which we briefly review here.

As usual, an *advice function* is a function $h : \mathbb{N} \to \{0,1\}^*$. Given a function $q : \mathbb{N} \to \mathbb{N}$, we write $\mathrm{ADV}(q)$ for the set of all advice functions $h$ such that $|h(n)| \leq q(n)$ for all $n \in \mathbb{N}$. Given a language $B$ and an advice function $h$, we define the language

$$B/h = \{x \in \{0,1\}^* \mid <x, h(|x|)> \in B\},$$

where $< \cdot\,, \cdot >$ is a standard string-pairing function, e.g., $< x,y >= 0^{|x|}1xy$. Given functions $t, q : \mathbb{N} \to N$, we define the advice class

$$\mathrm{DTIME}(t)/\mathrm{ADV}(q) = \{B/h \mid B \in \mathrm{DTIME}(t) \text{ and } h \in \mathrm{ADV}(q)\}.$$

**Definition** (Lutz and Mayordomo[10], Lutz[11]) *For $t, q, \nu : \mathbb{N} \to \mathbb{N}$, a language $A$ is* weakly $(t, q, \nu)$-stochastic *if, for all $B, C \in \mathrm{DTIME}(t)/\mathrm{ADV}(q)$ such that $|C_{=n}| \geq \nu(n)$ for all sufficiently large $n$,*

$$\lim_{n\to\infty} \frac{|(A \triangle B) \cap C_{=n}|}{|C_{=n}|} = \frac{1}{2}.$$

We write $\mathrm{WS}(t, q, \nu)$ for the set of all weakly $(t, q, \nu)$-stochastic languages.

The following result resembles the weak stochasticity theorems proved by Lutz and Mayordomo [10] and Lutz [11], but gives a more careful upper bound on the time complexity of the martingale.

**Theorem 2.4** (Weak Stochasticity Theorem) *Assume that $\alpha, \beta, \gamma, \tau \in \mathbb{R}$ satisfy $\alpha \geq 1, \beta \geq 1, \gamma > 0$, and $\tau > \alpha\beta$. Then there is an exact $2^{(\log n)^\tau}$-martingale $d$ such that*

$$S^\infty[d] \cup \mathrm{WS}(2^{n^\alpha}, n^\beta, 2^{\gamma n}) = \mathbf{C}.$$

**Proof.** Assume the hypothesis, and assume without loss of generality that $\alpha, \beta, \gamma, \tau \in \mathbb{Q}$. Fix $\alpha', \tau', \tau'' \in Q$ such that $\alpha < \alpha'$ and $\alpha'\beta < \tau'' < \tau' < \tau$. Let $U \in \mathrm{DTIME}(2^{n^{\alpha'}})$ be a language that is universal for $\mathrm{DTIME}(2^{n^\alpha}) \times \mathrm{DTIME}(2^{n^\alpha})$ in the following sense. For each $i \in \mathbb{N}$, let

$$C_i = \{x \in \{0,1\}^* | < s_i, 0x > \in U\},$$

$$D_i = \{x \in \{0,1\}^* | < s_i, 1x > \in U\}.$$

Then $\mathrm{DTIME}(2^{n^\alpha}) \times \mathrm{DTIME}(2^{n^\alpha}) = \{(C_i, D_i) | i \in \mathbb{N}\}$.

Define a function $d' : \mathbb{N}^3 \times \{0,1\}^* \to \mathbb{Q} \cap [0, \infty)$ as follows. If $k$ is not a power of 2, then $d'_{i,j,k}(w) = 0$. Otherwise, if $k = 2^n$, where $n \in \mathbb{N}$, then

$$d'_{i,j,k}(w) = \sum_{y,z \in \{0,1\}^{\leq n^\beta}} \Pr(Y_{i,j,k,y,z} | \mathbf{C}_w),$$

where the sets $Y_{i,j,k,y,z}$ are defined as follows. If $|(C_i/y)_{=n}| < 2^{\gamma n}$, then $Y_{i,j,k,y,z} = \emptyset$. If $|(C_i/y)_{=n}| \geq 2^{\gamma n}$, then $Y_{i,j,k,y,z}$ is the set of all $A \in \mathbf{C}$ such that

$$\left| \frac{|(A \triangle (D_i/z)) \cap (C_i/y)_{=n}|}{|(C_i/y)_{=n}|} - \frac{1}{2} \right| \geq \frac{1}{j+1}.$$

The definition of conditional probability immediately implies that, for each $i, j, k \in \mathbb{N}$, the function $d'_{i,j,k}$ is a martingale. Since $U \in \mathrm{DTIME}(2^{n^{\alpha'}})$ and $\alpha'\beta < \tau''$, the time required to compute each $\Pr(Y_{i,j,k,y,z} | \mathbf{C}_w)$ using binomial coefficients is at most $O(2^{(\log(i+j+k))^{\tau''}})$ steps, so the time required to compute $d'_{i,j,k}(w)$ is at most $O((2^{n^\beta} + 1)^2 \cdot 2^{(\log(i+j+k))^{\tau''}}) = O(2^{(\log(i+j+k))^{\tau'}})$ steps. Thus $d'$ is exactly $2^{(\log n)^{\tau'}}$-time-computable.

As in [10] and [11], the Chernoff bound tells us that, for all $i, j, n \in \mathbb{N}$ and $y, z \in \{0,1\}^{\leq n^\beta}$, writing $k = 2^n$,

$$\Pr(Y_{i,j,k,y,z}) \leq 2e^{-k^\gamma/2(j+1)^2},$$

whence

$$\begin{aligned} d'_{i,j,k}(\lambda) &\leq (2^{n^\beta}+1)^2 \cdot 2e^{-k^\gamma/2(j+1)^2} \\ &< e^{2n^\beta+3-k^\gamma/2(j+1)^2}. \end{aligned}$$

Let $a = \lceil \frac{1}{\gamma} \rceil$, let $\epsilon = \frac{\gamma}{4}$, and fix $k_0 \in \mathbb{N}$ such that

$$k^{2\epsilon} > k^\epsilon + 2(\log k)^\beta + 3$$

for all $k \geq k_0$. Define $g : \mathbb{N} \to \mathbb{N}$ by

$$g(j) = 4^a(j+1)^{4a} + k_0$$

for all $j \in \mathbb{N}$. Then $g$ is a polynomial and, for all $i, j, n \in \mathbb{N}$, writing $k = 2^n$,

$$\begin{aligned} k \geq g(j) \quad &\Rightarrow \quad \left\{ \begin{array}{lcl} k^\gamma &=& k^{2\epsilon}k^{2\epsilon} \\ &>& [4^a(j+1)^{4a}]^{2\epsilon}(k^\epsilon + 2(\log k)^\beta + 3) \\ &\geq& 2(j+1)^2(k^\epsilon + 2n^\beta + 3) \end{array} \right. \\ &\Rightarrow \quad d'_{i,j,k}(\lambda) < e^{-k^\epsilon}. \end{aligned}$$

It follows by Lemma 2.2 that the series $\sum_{k=0}^{\infty} d'_{i,j,k}(\lambda)$, for $i, j \in \mathbb{N}$, are uniformly p-convergent. Since $1 < \tau' < \tau$, it follows by Theorem 2.3 that there is an exact $2^{(\log n)^\tau}$-martingale $d$ such that

$$\bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} \bigcap_{t=0}^{\infty} \bigcup_{k=t}^{\infty} S^1[d'_{i,j,k}] \subseteq S^\infty[d]. \tag{19}$$

Now assume that $A \notin \mathrm{WS}(2^{n^\alpha}, n^\beta, 2^{\gamma n})$. Then, by the definition of weak stochasticity, we can fix $i, j \in \mathbb{N}$, functions $h_1, h_2 \in \mathrm{ADV}(n^\beta)$, and an infinite set $J \subseteq \mathbb{N}$ such that, for all $n \in J$, $A \in Y_{i,j,k,h_1(n),h_2(n)}$, where $k = 2^n$. For each $n \in J$, then, there is a prefix $w \sqsubseteq A$ such that $\mathbf{C}_w \subseteq Y_{i,j,k,h_1(n),h_2}(n)$, whence

$$d'_{i,j,k}(w) \geq \Pr(Y_{i,j,k,h_1(n),h_2(n)} | \mathbf{C}_w) = 1,$$

i.e., $A \in S^1[d'_{i,j,k}]$. This argument shows that

$$\bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} \bigcap_{t=0}^{\infty} \bigcup_{k=t}^{\infty} S^1[d'_{i,j,k}] \cup \mathrm{WS}(2^{n^\alpha}, n^\beta, 2^{\gamma n}) = \mathbf{C}.$$

It follows by (19) that

$$S^\infty[d] \cup \mathrm{WS}(2^{n^\alpha}, n^\beta, 2^{\gamma n}) = \mathbf{C}. \qquad \square$$

The technique of martingale dilation was introduced by Ambos-Spies, Terwijn, and Zheng [3]. It has also been used by Juedes and Lutz[8] and generalized considerably by Breutzmann and Lutz [6]. We use the notation of [8] here.

The *restriction* of a string $w = b_0 b_1 \cdots b_{n-1} \in \{0,1\}^*$ to a language $A \subseteq \{0,1\}^*$ is the string $w \upharpoonright A$ obtained by concatenating the successive bits $b_i$ for which $s_i \in A$. If $f : \{0,1\}^* \to \{0,1\}^*$ is strictly increasing and $d$ is a martingale, then the *f-dilation* of $d$ is the function $f\hat{\ }d : \{0,1\}^* \to [0,\infty)$ defined by

$$f\hat{\ }d(w) = d(w \upharpoonright range(f))$$

for all $w \in \{0,1\}^*$.

**Lemma 2.5** (Martingale Dilation Lemma - Ambos-Spies, Terwijn, and Zheng[3]) *If $f : \{0,1\}^* \to \{0,1\}^*$ is strictly increasing and $d$ is a martingale, then $f\hat{\ }d$ is also a martingale. Moreover, for every language $A \in \{0,1\}^*$, if $d$ succeeds on $f^{-1}(A)$, then $f\hat{\ }d$ succeeds on $A$.*

Finally, we summarize the most basic ideas of resource-bounded measure in E and $\mathrm{E}_2$. A p-*martingale* is a martingale that is, for some $k \in \mathbb{N}$, an $n^k$-martingale. A $\mathrm{p}_2$-*martingale* is a martingale that is, for some $k \in \mathbb{N}$, a $2^{(\log n)^k}$-martingale.

**Definition** (Lutz [9])

1. *A set $X$ of languages has* p-measure 0, *and we write $\mu_{\mathrm{p}}(X) = 0$, if there is a* p-*martingale $d$ such that $X \subseteq S^\infty[d]$.*

2. *A set $X$ of languages has* $\mathrm{p}_2$-measure 0, *and we write $\mu_{\mathrm{p}_2}(X) = 0$, if there is a* $\mathrm{p}_2$-*martingale $d$ such that $X \subseteq S^\infty[d]$.*

3. *A set $X$ of languages has* measure 0 *in* E, *and we write $\mu(X|\mathrm{E}) = 0$, if $\mu_{\mathrm{p}}(X \cap \mathrm{E}) = 0$.*

4. *A set $X$ of languages has* measure 0 *in* $\mathrm{E}_2$, *and we write $\mu(X|\mathrm{E}_2) = 0$, if $\mu_{\mathrm{p}_2}(X \cap \mathrm{E}_2) = 0$.*

5. *A set $X$ of languages has* measure 1 *in* E, *and we write $\mu(X|\mathrm{E}) = 1$, if $\mu(X^c|\mathrm{E}) = 0$. In this case, we say that $X$ contains* almost every *element of* E.

6. *A set $X$ of languages has* measure 1 *in* $\mathrm{E}_2$, *and we write $\mu(X|\mathrm{E}_2) = 1$, if $\mu(X^c|\mathrm{E}_2) = 0$. In this case, we say that $X$ contains* almost every *element of* $\mathrm{E}_2$.

7. *The expression $\mu(X|\mathrm{E}) \neq 0$ means that $X$ does* not *have measure 0 in* E. *Note that this does* not *assert that "$\mu(X|\mathrm{E})$" has some nonzero value. Similarly, the expression $\mu(X|\mathrm{E}_2) \neq 0$ means that $X$ does not have measure 0 in* $\mathrm{E}_2$.

It is shown in [9] that these definitions endow E and $\mathrm{E}_2$ with internal measure structure. This structure justifies the intuition that, if $\mu(X|\mathrm{E}) = 0$, then $X \cap \mathrm{E}$ is a *negligibly small* subset of E (and similarly for $\mathrm{E}_2$).

## 3 Results

The key to our main theorem is the following lemma, which says that languages that are $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{T}}$-reducible to non-dense languages cannot be very stochastic.

**Lemma 3.1** (Main Lemma) *For all real numbers $\alpha < 1$ and $\beta > 1 + \alpha$,*

$$\mathrm{P}_{n^\alpha-\mathrm{T}}(\mathrm{DENSE}^c) \cap \mathrm{WS}(2^n, n^\beta, 2^{\frac{n}{2}}) = \emptyset.$$

**Proof.** Let $\alpha < 1$ and $\beta > 1+\alpha$, and assume without loss of generality that $\alpha$ and $\beta$ are rational. Let $A \in \mathrm{P}_{n^\alpha-\mathrm{T}}(\mathrm{DENSE}^c)$. It suffices to show that $A$ is not weakly $(2^n, n^\beta, 2^{\frac{n}{2}})$-stochastic.

Since $A \in \mathrm{P}_{n^\alpha-\mathrm{T}}(\mathrm{DENSE}^c)$, there exist a non-dense language $S$, a polynomial $q(n)$, and a $q(n)$-time-bounded oracle Turing machine $M$ such that $A = L(M^S)$ and, for every $x \in \{0,1\}^*$ and $B \subseteq \{0,1\}^*$, $M$ makes exactly $\lfloor |x|^\alpha \rfloor$queries (all distinct) on input $x$ with oracle $B$. Call these queries $Q^B(x,1), \ldots, Q^B(x, \lfloor |x|^\alpha \rfloor)$ in the order in which $M$ makes them.

For each $B \in \{0,1\}^*$ and $n \in \mathbb{N}$, define an equivalence relation $\approx_{B,n}$ on $\{0,1\}^{\leq q(n)}$ by

$$u \approx_{B,n} v \Leftrightarrow (\forall w)[u \leq w \leq v \Rightarrow [\![w \in B]\!] = [\![u \in B]\!]]$$

and an equivalence relation $\equiv_{B,n}$ on $\{0,1\}^n$ by

$$x \equiv_{B,n} y \Leftrightarrow (\forall i)[1 \leq i \leq n^\alpha \Rightarrow Q^B(x,i) \approx_{B,n} Q^B(y,i)].$$

Note that $\approx_{B,n}$ has at most $2|B_{\leq q(n)}|+1$ equivalence classes, so $\equiv_{B,n}$ has at most $(2|B_{\leq q(n)}|+1)^{n^\alpha}$ equivalence classes.

Let $\epsilon = \frac{1-\alpha}{2}$, and let $J$ be the set of all $n \in \mathbb{N}$ for which the following three conditions hold.

(i) $2|S_{\leq q(n)}| + 1 \leq 2^{n^\epsilon}$.

(ii) $n^{\alpha+\epsilon} \leq \frac{n}{2}$.

(iii) $n^\alpha(2n+1) \leq n^\beta$.

Since $\alpha + \epsilon < 1$ and $\beta > 1 + \alpha$, conditions (ii) and (iii) hold for all sufficiently large $n$. Since $\epsilon > 0$ and $S$ is not dense, condition (i) holds for infinitely many $n$. Thus the set $J$ is infinite.

Define an advice function $h : \mathbb{N} \to \{0,1\}^*$ as follows. If $n \notin J$, then $h(n) = \lambda$. If $n \in J$, then let $D_n$ be a maximum-cardinality equivalence class of the relation $\equiv_{S,n}$. For each $1 \leq i \leq \lfloor n^\alpha \rfloor$, fix strings $y_{n,i}, z_{n,i} \in D_n$ such that, for all $x \in D_n$,

$$Q^S(y_{n,i}, i) \leq Q^S(x,i) \leq Q^S(z_{n,i}, i).$$

Let

$$\begin{aligned} h_1(n) &= y_{n,1} \cdots y_{n,\lfloor n^\alpha \rfloor}, \\ h_2(n) &= z_{n,1} \cdots z_{n,\lfloor n^\alpha \rfloor}, \\ h_3(n) &= [\![Q^S(y_{n,1},1) \in S]\!] \cdots [\![Q^S(y_{n,\lfloor n^\alpha \rfloor}, \lfloor n^\alpha \rfloor) \in S]\!], \\ h(n) &= h_1(n)h_2(n)h_3(n). \end{aligned}$$

Note that $|h(n)| = \lfloor n^\alpha \rfloor (2n+1) \le n^\beta$ for all $n \in J$, so $h \in \mathrm{ADV}(n^\beta)$.

For each $n \in \mathbb{N}$, let $t = \lfloor n^\alpha \rfloor$, and let $C_n$ be the set of all coded pairs

$$< x, y_1 \cdots y_t z_1 \cdots z_t b_1 \cdots b_t >$$

such that $x, y_1, \ldots, y_t, z_1, \ldots, z_t \in \{0,1\}^n$, $b_1, \ldots, b_t \in \{0,1\}$, and, for each $1 \le i \le t$,

$$Q^{b_1 \cdots b_t}(y_i, i) \le Q^{b_1 \cdots b_t}(x, i) \le Q^{b_1 \cdots b_t}(z_i, i),$$

where $Q^{b_1 \cdots b_t}(w, i)$ denotes the $i$th query of $M$ on input $w$ when the successive oracle answers are $b_1, \ldots, b_t$. Let $B_n$ be the set of all such coded pairs in $C_n$ such that $M$ accepts on input $x$ when the successive oracle answers are $b_1, \ldots, b_t$. Finally, define the languages

$$B = \{< x, v > \mid v = \lambda \text{ or } < x, v > \in B_{|x|}\},$$

$$C = \{< x, v > \mid v = \lambda \text{ or } < x, v > \in C_{|x|}\}.$$

It is clear that $B, C \in \mathrm{DTIME}(2^n)$. Also, by our construction of these sets and the advice function $h$, for each $n \in \mathbb{N}$, we have

$$(C/h)_{=n} = \begin{cases} D_n & \text{if } n \in J \\ \{0,1\}^n & \text{if } n \notin J \end{cases}$$

and

$$(B/h)_{=n} = \begin{cases} A \cap D_n & \text{if } n \in J \\ \{0,1\}^n & \text{if } n \notin J \end{cases}.$$

For each $n \in J$, if $\kappa(n)$ is the number of equivalence classes of $\equiv_{S,n}$, then

$$\kappa(n) \le (2|S_{\le q(n)}| + 1)^{n^\alpha} \le (2^{n^\epsilon})^{n^\alpha} = 2^{n^{\alpha+\epsilon}},$$

so

$$|D_n| \ge \frac{2^n}{\kappa(n)} \ge 2^{n - n^{\alpha+\epsilon}} \ge 2^{\frac{n}{2}}.$$

It follows that $|(C/h)_{=n}| \ge 2^{\frac{n}{2}}$ for all $n \in \mathbb{N}$.

Finally, for all $n \in J$,

$$(A \bigtriangleup (B/h)) \cap (C/h)_{=n} = (A \bigtriangleup (A \cap D_n)) \cap D_n = \emptyset.$$

Since $J$ is infinite, it follows that

$$\frac{|(A \bigtriangleup (B/h)) \cap (C/h)_{=n}|}{|(C/h)_{=n}|} \not\to \frac{1}{2}$$

as $n \to \infty$. Since $B, C \in \mathrm{DTIME}(2^n)$, $h \in \mathrm{ADV}(n^\beta)$, and $|C_{=n}| \ge 2^{n^2}$ for all $n \in \mathbb{N}$, this shows that $A$ is not weakly $(2^n, n^\beta, 2^{\frac{n}{2}})$-stochastic. □

We now prove our main result.

**Theorem 3.2** (Main Theorem) *For every real number* $\alpha < 1$,

$$\mu_{\rm p}(\mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c)) = \mu_{\rm p_2}(\mathrm{P}_{n^{\alpha}-\mathrm{T}}(\mathrm{DENSE}^c)) = 0.$$

**Proof.** Let $\alpha < 1$, and let $\beta = \frac{3+\alpha}{2}$, so that $1 + \alpha < \beta < 2$. By Theorem 2.4, there is an exact $2^{(\log n)^2}$-martingale $d$ such that

$$S^\infty[d] \cup \mathrm{WS}(2^n, n^\beta, 2^{\frac{n}{2}}) = \mathbf{C}.$$

By Lemma 3.1, we then have

$$\mathrm{P}_{n^\alpha-\mathrm{T}}(\mathrm{DENSE}^c) \subseteq S^\infty[d].$$

Since $d$ is a $\rm p_2$-martingale, this implies that $\mu_{\rm p_2}(\mathrm{P}_{n^\alpha-\mathrm{T}}(\mathrm{DENSE}^c)) = 0$.

Define $f : \{0,1\}^* \to \{0,1\}^*$ by

$$f(x) = 0^{|x|^2-|x|-1}1x.$$

Then $f$ is strictly increasing, so $f\hat{\ }d$, the $f$-dilation of $d$, is a martingale. The time required to compute $f\hat{\ }d(w)$ is

$$O(|w|^2 + 2^{(\log|w'|)^2})$$

steps, where $w' = w{\restriction}range(f)$. (This allows $O(|w|^2)$ steps to compute $w'$ and then $O(2^{(\log|w'|)^2})$ steps to compute $d(w')$.)

Now $|w'|$ is bounded above by the number of strings $x$ such that $|x|^2 \leq |s_{|w|}| = \lfloor \log(1+|w|) \rfloor$, so

$$|w'| < 2^{1+\sqrt{\log(1+|w|)}}.$$

Thus the time required to compute $f\hat{\ }d(w)$ is

$$O(|w|^2 + 2^{(1+\sqrt{\log(1+|w|)}\,)^2}) = O(|w|^2)$$

steps, so $f\hat{\ }d$ is an $n^2$-martingale.

Now let $A \in \mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c)$. Then $f^{-1}(A) \in \mathrm{P}_{n^\alpha-\mathrm{T}}(\mathrm{DENSE}^c) \subseteq S^\infty[d]$, so $A \in S^\infty[f\hat{\ }d]$ by Lemma 2.5. This shows that $\mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c) \subseteq S^\infty[f\hat{\ }d]$. Since $f\hat{\ }d$ is an $n^2$-martingale, it follows that $\mu_{\rm p}(\mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c)) = 0$. □

We now develop a few consequences of the Main Theorem. The first is immediate.

**Corollary 3.3** *For every real number* $\alpha < 1$,

$$\mu(\mathrm{P}_{n^{\alpha/2}-\mathrm{T}}(\mathrm{DENSE}^c) \mid \mathrm{E}) = \mu(\mathrm{P}_{n^\alpha-\mathrm{T}}(\mathrm{DENSE}^c) \mid \mathrm{E}_2) = 0.$$

The following result on the density of weakly complete (or weakly hard) languages now follows immediately from Corollary 3.3.

**Corollary 3.4** *For every real number* $\alpha < 1$, *every language that is weakly* $\leq^{\mathrm{P}}_{n^{\alpha/2}-\mathrm{T}}$*-hard for* E *or weakly* $\leq^{\mathrm{P}}_{n^\alpha-\mathrm{T}}$*-hard for* $\mathrm{E}_2$ *is dense.*

Our final two corollaries concern consequences of the strong hypotheses $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$ and $\mu_{\mathrm{p}_2}(\mathrm{NP}) \neq 0$. The relative strengths of these hypotheses are indicated by the known implications

$$\mu(\mathrm{NP} \mid \mathrm{E}) \neq 0 \Rightarrow \mu(\mathrm{NP} \mid \mathrm{E}_2) \neq 0 \Leftrightarrow \mu_{\mathrm{p}_2}(\mathrm{NP}) \neq 0 \Rightarrow \mu_{\mathrm{p}}(\mathrm{NP}) \neq 0 \Rightarrow \mathrm{P} \neq \mathrm{NP}.$$

(The leftmost implication was proven by Juedes and Lutz[8]. The remaining implications follow immediately from elementary properties of resource-bounded measure.)

**Corollary 3.5** *Let* $\alpha < 1$. *If* $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, *then every language that is* $\leq^{\mathrm{P}}_{n^{\alpha/2}-\mathrm{T}}$*-hard for* NP *is dense. If* $\mu_{\mathrm{p}_2}(\mathrm{NP}) \neq 0$, *then every language that is* $\leq^{\mathrm{P}}_{n^{\alpha}-\mathrm{T}}$*-hard for* NP *is dense.*

We conclude by considering the densities of languages to which SAT can be adaptively reduced.

**Definition** *A function* $g : \mathbb{N} \to \mathbb{N}$ *is* subradical *if* $\log g(n) = o(\log n)$.

It is easy to see that a function $g$ is subradical if and only if, for all $k > 0$, $g(n) = o(\sqrt[k]{n})$. (This is the reason for the name "subradical.") Subradical functions include very slow-growing functions such as $\log n$ and $(\log n)^5$, as well as more rapidly growing functions such as $2^{(\log n)^{0.99}}$.

**Corollary 3.6** *If* $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, $g : \mathbb{N} \to \mathbb{N}$ *is subradical, and SAT* $\leq^{\mathrm{P}}_{g(n)-\mathrm{T}} H$, *then* $H$ *is dense.*

**Proof.** Assume the hypothesis. Let $A \in \mathrm{NP}$. Then there is a $\leq^{\mathrm{P}}_{\mathrm{m}}$-reduction $f$ of $A$ to SAT. Fix a polynomial $q(n)$ such that, for all $x \in \{0,1\}^*$, $|f(x)| \leq q(|x|)$. Composing $f$ with the $\leq^{\mathrm{P}}_{g(n)-\mathrm{T}}$-reduction of SAT to $H$ that we have assumed to exist then gives a $\leq^{\mathrm{P}}_{g(q(n))-\mathrm{T}}$-reduction of $A$ to $H$. Since $g$ is subradical, $\log g(q(n)) = o(\log q(n)) = o(\log n)$, so for all sufficiently large $n$, $g(q(n)) \leq 2^{\frac{\log n}{4}} = n^{\frac{1}{4}}$. Thus $A \leq^{\mathrm{P}}_{n^{\frac{1}{4}}-\mathrm{T}} H$.

The above argument shows that $H$ is $\leq^{\mathrm{P}}_{n^{\frac{1}{4}}-\mathrm{T}}$-hard for NP. Since we have assumed $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, it follows by Corollary 3.5 that $H$ is dense. □

To put the matter differently, Corollary 3.6 tells us that if SAT is polynomial-time reducible to a non-dense language with at most $2^{(\log n)^{0.99}}$ adaptive queries, then NP has measure 0 in E and in $\mathrm{E}_2$.

# 4 Questions

As noted in the introduction, the relationships between weak hardness notions for E and $\mathrm{E}_2$ under reducibilities such as $\leq^{\mathrm{P}}_{\mathrm{T}}$, $\leq^{\mathrm{P}}_{n^{\alpha}-\mathrm{T}}$, and $\leq^{\mathrm{P}}_{n^{\alpha}-\mathrm{tt}}$ remain to be resolved. Our main theorem also leaves open the question whether $\leq^{\mathrm{P}}_{n^{\alpha}-\mathrm{T}}$-hard languages for E must be dense when $\frac{1}{2} \leq \alpha < 1$. We are in the curious situation of knowing that the classes $\mathrm{P}_{n^{0.99}-\mathrm{tt}}(\mathrm{DENSE}^c)$ and $\mathrm{P}_{n^{0.49}-\mathrm{T}}(\mathrm{DENSE}^c)$ have p-measure 0, but not knowing whether $\mathrm{P}_{n^{0.50}-\mathrm{T}}(\mathrm{DENSE}^c)$ has p-measure 0. Indeed, at this time we cannot even prove that $\mathrm{E} \not\subseteq \mathrm{P}_{n^{0.50}-\mathrm{T}}(\mathrm{SPARSE})$. Further progress on this matter would be illuminating.

## References


[1] K. Ambos-Spies, Randomness, relativizations, and polynomial reducibilities. In *Proceedings of the First Structure in Complexity Theory Conference*, pp. 23-34. Springer-Verlag, 1986.

[2] K. Ambos-Spies and E. Mayordomo, Resource-bounded measure and randomness, in A. Sorbi(ed.), *Complexity, Logic, and Recursion Theory*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, to appear.

[3] K. Ambos-Spies, S. A. Terwijn, and X. Zheng, Resource-bounded randomness and weakly complete problems. *Theoretical Computer Science*. To appear.

[4] C. H. Bennett and J. Gill, Relative to a random oracle $A$, $\mathrm{P}^A \neq \mathrm{NP} \neq co - \mathrm{NP}^A$ with probability 1, *SIAM Journal on Computing* 10 (1981), pp.96-113.

[5] L. Berman and J. Hartmanis, On isomorphism and density of NP and other complete sets. *SIAM Journal on Computing*, 6:305-322, 1977.

[6] J. M. Breutzmann and J. H. Lutz, Equivalence of measures of complexity classes, *Proceedings of the Fourteenth Symposium on Theoretical Aspects of Computer Science*, Springer-Verlag, 1997, to appear.

[7] B. Fu, With quasi-linear queries, EXP is not polynomial time Turing reducible to sparse sets. *SIAM Journal on Computing*, 24:1082-1090, 1995.

[8] D. W. Juedes and J. H. Lutz, Weak completeness in E and $\mathrm{E}_2$. *Theoretical Computer Science*, 143:149-158, 1995.

[9] J. H. Lutz, Almost everywhere high nonuniform complexity, *Journal of Computer and System Sciences*, 19:1100-1131, 1990.

[10] J. H. Lutz and E. Mayordomo, Measure stochasticity, and the density of hard languages. *SIAM Journal on Computing*, 23:762-779, 1994.

[11] J. H. Lutz, One-way functions and balanced NP. *Theoretical Computer Science*, to appear.

[12] J. H. Lutz, The quantitative structure of exponential time, in L. A. Hemaspaandra and A. L. Selman (eds.), *Complexity Theory Retrospective II*, Springer-Verlag, 1996, to appear. A preliminary version appears in *Proceedings of The Eighth Annual Structure in Complexity Theory Conference* (San Diego, CA, May 18-21, 1993), IEEE Computer Society Press, 1993, pp. 158-175.

[13] S. R. Mahaney, Sparse complete sets for NP: Solution of a conjecture of Berman and Hartmanis. *Journal of Computer and System Sciences*, 25:130-143, 1982.

[14] E. Mayordomo, Contributions to the Study of Resource-Bounded Measure. PhD thesis, Universitat Politècnica de Catalunya, Barcelona, Spain, 1994.

[15] A. R. Meyer, 1977. Reported in [5].

[16] M. Ogiwara and O. Watanabe, On polynomial bounded truth-table reducibility of NP sets to sparse sets. *SIAM Journal on Computing*, 20(3):471-483, June 1991.

[17] O. Watanabe, On the Structure of Intractable Complexity Classes. PhD thesis, Tokyo Institute of Technology, 1987.