Dimension spectra of random subfractals of self-similar fractals

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Abstract

The (constructive Hausdorff) dimension of a point x in Euclidean space is the *algorithmic information density* of x. Roughly speaking, this is the least real number $\dim(x)$ such that $r \times \dim(x)$ bits suffices to specify x on a general-purpose computer with arbitrarily high precisions 2^{-r}. The *dimension spectrum* of a set X in Euclidean space is the subset of $[0, n]$ consisting of the dimensions of all points in X.

The dimensions of points have been shown to be geometrically meaningful (Lutz 2003, Hitchcock 2003), and the dimensions of points in self-similar fractals have been completely analyzed (Lutz and Mayordomo 2008). Here we begin the more challenging task of analyzing the dimensions of points in random fractals. We focus on fractals that are randomly selected subfractals of a given self-similar fractal. We formulate the specification of a point in such a subfractal as the outcome of an infinite two-player game between a *selector* that selects the subfractal and a *coder* that selects a point within the subfractal. Our selectors are algorithmically random with respect to various probability measures, so our selector-coder games are, from the coder's point of view, games against nature.

We determine the dimension spectra of a wide class of such randomly selected subfractals. We show that each such fractal has a dimension spectrum that is a closed interval whose endpoints can be computed or approximated from the parameters of the fractal. In general, the maximum of the spectrum is determined by the degree to which the coder can *reinforce* the randomness in the selector, while the minimum is determined by the degree to which the coder can *cancel* randomness in the selector. This constructive and destructive interference between the players' randomnesses is somewhat subtle, even in the simplest cases. Our proof techniques include van Lambalgen's theorem on independent random sequences, measure preserving transformations, an application of network flow theory, a Kolmogorov complexity lower bound argument, and a nonconstructive proof that this bound is tight.

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1 Introduction

Fractals are inherently information-theoretic objects. The dimension n of a Euclidean space $\mathbb{R}^n$ is a measure of the amount of information (number of real numbers) that suffices to specify a point in $\mathbb{R}^n$ in a natural way. Similarly, the fact that the Hausdorff dimension of the Cantor "middle-thirds" set C is $\dim_H(C) = \log 2/\log 3 \approx 0.63$ tells us that it only takes about 0.63 of a real number to specify a point in C in a natural way. That is, roughly $(0.63)r$ bits suffice to specify the first r bits of a point in C. Intuitively, then, the Hausdorff (fractal) dimension $\dim_H(C)$ is an upper bound on the "information densities" of points in the fractal C.

Of course some points in the Cantor set can be specified even more concisely. The theory of constructive dimension, a computability-theoretic extension of Hausdorff dimension developed in the present century [16], assigns each *individual point* x in a Euclidean space $\mathbb{R}^n$ a *dimension* $\dim(x) \in [0, n]$ that is a measure of its information density. This notion of dimension has been shown to be geometrically meaningful. For example, if $X \subseteq \mathbb{R}^n$ is a "reasonably simple" set, in the sense that X is a union of Π^0_1 (i.e., computably closed) sets, then

$$\dim_H(X) = \sup_{x\in X} \dim(x), \tag{1.1}$$

which is a nonclassical, pointwise characterization of the classical Hausdorff dimensions of such sets [16, 12].

The *self-similar fractals* form the best known and best understood class of fractals. (See section 2.3 for a detailed review of self-similar fractals.) Each self-similar fractal F is given by an *iterated function system (IFS)* $S = (S_1, \ldots, S_{m-1})$ of *contracting similarities* S_i. A celebrated theorem of Moran [20] states that

$$\dim_H(F) = \mathrm{sdim}(F) \tag{1.2}$$

holds for every self-similar fractal F, where $\mathrm{sdim}(F)$ is the *similarity dimension* of F. Much of the importance of this theorem arises from the fact that $\mathrm{sdim}(F)$ is easy to compute from the contraction ratios $c_0, \ldots, c_{m-1}$ of the respective similarities $S_1, \ldots, S_{m-1}$. That is, (1.2) gives an easy way to compute the Hausdorff dimensions of self-similar fractals.

The dimensions of points in computably self-similar fractals (those for which $S_1, \ldots, S_{m-1}$ are computable) have now been completely analyzed. If F is a self-similar fractal as above, then each point $x \in F$ is naturally given by at least one *coding sequence* $U \in \Sigma_m^\infty$, where $\Sigma_m = \{0, \ldots, m-1\}$. Intuitively, x is the result of a limiting process in which, at each stage $t \in \mathbb{N}$, we apply the contracting similarity $S_{U[t]}$. The main theorem of [18] says that, if F is computably self-similar, then, for each $x \in F$ and each coding sequence U for x,

$$\dim(x) = \mathrm{sdim}(F)\dim^{\pi_S}(U), \tag{1.3}$$

where $\dim^{\pi_S}(U) \in [0, 1]$ is the dimension of the sequence $U \in \Sigma_m^\infty$ with respect to the *similarity probability measure* π_S on Σ_m, which arises from the IFS S in a natural manner. (This is a constructive version of Billingsley dimension [3] introduced in [18].)

This paper begins the more challenging task of analyzing the dimensions of points in random fractals. We focus on a particular class of random fractals, the *random subfractals* of self-similar fractals. For a concrete example, let F be the Sierpinski triangle. This is a well-known self-similar fractal. Intuitively, consider a *selector* σ that randomly chooses just two of the three top-level subtriangles of F, then randomly chooses just two subtriangles of each of these, etc., ultimately obtaining a subfractal F_σ of F. If σ is algorithmically random (with respect to some probability distribution), then we call F_σ a *random subfractal* of F. An individual element $x \in F_\sigma$ is specified by a *coder* $T \in \{0,1\}^\infty$ that tells us, at successive stages, *which* of the two selected subtriangles has x as an element. This interplay between σ and T is formalized in general terms as the *selector-coder game* in section 3. Since our selectors are all random, our coders are playing games against nature.

The *dimension spectrum* of a set $X \subseteq \mathbb{R}^n$ is the set $\mathrm{sp}(X) \subseteq [0,n]$ consisting of all $\dim(x)$ for $x \in X$. Our objective is to determine the dimension spectrum $\mathrm{sp}(F_\sigma)$ of random subfractals of a given computably self-similar fractal F. By (1.3) and the fact that $\dim^{\pi_S}(U)$ takes on all values in $[0,1]$, we have

$$\mathrm{sp}(F_\sigma) \subseteq \mathrm{sp}(F) = [0, \mathrm{sdim}(F)] \tag{1.4}$$

in any case.

Our main theorem, Theorem 4.1, concerns *similarity-random subfractals* F_σ of a given self-similar fractal F, i.e., subfractals of F specified by a selector σ that is algorithmically random with respect to a natural extension of the above-mentioned similarity probability measure π_S on Σ_m. This theorem says that each such $\mathrm{sp}(F_\sigma)$ is an interval containing $\mathrm{sdim}(F)$, and it gives upper and lower bounds on the left endpoint of the interval $\mathrm{sp}(F_\sigma)$. In the particular case where the contraction ratios $c_0, \ldots, c_{m-1}$ are all the same, these upper and lower bounds coincide, and our main theorem gives the exact dimension spectrum of F_σ.

Intuitively, the proof that $\dim(F) \in \mathrm{sp}(F_\sigma)$ is carried out by showing that the coder T can *reinforce* the randomness in σ, while the bounds on the left endpoint of $\mathrm{sp}(F_\sigma)$ quantify the degree to which the coder T can *cancel* some of the randomness of σ. This constructive and destructive interference between the players' randomnesses is somewhat subtle, and the proof of our main theorem reflects this, using van Lambalgen's theorem on independent random sequences, measure-preserving transformations, a Kolmogorov complexity lower bound argument, and a nonconstructive proof that this lower bound is nearly (and, in the single-contraction-ratio case, exactly) tight.

In section 5 we give results on the dimension spectra of subfractals of self-similar fractals that are random with respect to more general probability measures. Our proofs use the above methods, together with network flow theory and the divergence formula for randomness and dimension [17].

The randomness cancellation phenomena that play such a large role here have also arisen in other contexts, notably dimension spectra of random closed sets [2, 7] and of random translations of the Cantor set [8]. Our work is as much an investigation of these phenomena as it is an analysis of a particular class of fractals.

2 Preliminaries

2.1 Notation and Terminology

Given a finite alphabet Σ, we write Σ^* for the set of all (finite) *strings* over Σ and Σ^∞ for the set of all (infinite) *sequences* over Σ. If $\psi \in \Sigma^* \cup \Sigma^\infty$ and $0 \leq i \leq j < |\psi|$, where $|\psi|$ is the length of ψ, then $\psi[i]$ is the ith symbol in ψ (where $\psi[0]$ is the leftmost symbol in ψ), and $\psi[i..j]$ is the string consisting of the ith through the jth symbols in ψ. If $w \in \Sigma^*$ and $\psi \in \Sigma^* \cup \Sigma^\infty$, then w is a prefix of ψ, and we write $w \sqsubseteq \psi$, if there exists $i \in \mathbb{N}$ such that $w = \psi[0..i-1]$. If $A \subseteq \Sigma^*$ then $A^{=n} = \{x \mid x \in A \wedge |x| = n\}$.

For $k \in \mathbb{N}$, $\Sigma_k = \{0, \ldots, k-1\}$. Let $s_0^{(k)}$, $s_1^{(k)}$, $s_2^{(k)}$, ... be the standard enumeration of Σ_k^*. For $w \in \Sigma_k^*$, $\mathrm{index}^{(k)}(w)$ is the index of w in the standard enumeration of Σ_k^*, i.e., $s^{(k)}_{\mathrm{index}^{(k)}(w)} = w$.

For a set A and $k \in \mathbb{N}$, $[A]^k$ is the set of all k-element subsets of A.

For functions on Euclidean space, we use the computability notion formulated by Grzegorczyk [11] and Lacombe [14] in the 1950's and exposited in the monographs by Pour-El and Richards [21], Ko [13], and Weihrauch [23] and in the recent survey paper by Braverman and Cook [5]. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *computable* if there is an oracle Turing machine M with the following property. For all $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, if M is given a function oracle $\varphi_x : \mathbb{N} \to \mathbb{Q}^n$ such that, for all $k \in \mathbb{N}$, $|\varphi_x(k) - x| \leq 2^{-k}$, then M, with oracle φ_x and input r, outputs a rational point $M^{\varphi_x}(r) \in \mathbb{Q}^n$ such that $|M^{\varphi_x}(r) - f(x)| \leq 2^{-r}$.

For subsets of Euclidean space, we use the computability notion introduced by Brattka and Weihrauch [4] (see also [23, 5]). A set $X \subseteq \mathbb{R}^n$ is computable if there is a computable function $f_X : \mathbb{Q}^n \times \mathbb{N} \to \{0, 1\}$ that satisfies the following two conditions for all $q \in \mathbb{Q}^n$ and $r \in \mathbb{N}$.

(i) If there exists $x \in X$ such that $|x - q| \leq 2^{-r}$, then $f_X(q, r) = 1$.

(ii) If there is no $x \in X$ such that $|x - q| \leq 2^{1-r}$, then $f_X(q, r) = 0$.

All logarithms in this paper are base-2. We use μ for the uniform probability measures for Cantor spaces with all finite alphabet sizes.

2.2 Randomness and Dimension

Recent advances in computability theory have yielded notions of dimension for single points in Euclidean spaces [16, 1, 18] (as opposed to the classical view where single points have dimension zero). These notions are robust in the sense that they admit several equivalent characterizations. We start with a description of constructive dimension on Σ_m^∞. For the rest of this section let π denote a positive probability measure on Σ_m, extended by product to Σ_m^*.

Definition. Let $s \in [0, \infty)$. A *π-s-gale* is a function $d : \Sigma_m^* \to [0, \infty)$ such that for all $w \in \Sigma_m^*$, $d(w)\pi^s(w) = \sum_{b \in \Sigma_m} d(wb)\pi^s(wb)$. A π-1-gale is called a *π-martingale.* A π-s-gale is *constructive* (*lower semicomputable*) if there exists a computable function $\hat{d} : \Sigma_m^* \times \mathbb{N} \to \mathbb{Q}$ such that

1. for all w, t, $\hat{d}(w, t) < d(w)$, and
2. for all w, $\lim_{t\to\infty} \hat{d}(w, t) = d(w)$.

A π-s-martingale *succeeds* on $T \in \Sigma_m^\infty$ if $\limsup\limits_{n\to\infty} d(T[0..n]) = \infty$. A π-s-martingale *succeeds strongly* on $T \in \Sigma_m^\infty$ if $\liminf\limits_{n\to\infty} d(T[0..n]) = \infty$.

For a π-s-gale d, define its *success set* by $S^\infty[d] = \{T \in \Sigma_m^\infty | \limsup\limits_{n\to\infty} d(T[0..n]) = \infty\}$ and its *strong success* set by $S^\infty_{\mathrm{str}}[d] = \{T \in \Sigma_m^\infty | \liminf\limits_{n\to\infty} d(T[0..n]) = \infty\}$.

A sequence $T \in \Sigma_m^\infty$ is *π-random* if no constructive π-martingale succeeds on it.

Definition. Let $X \subseteq \Sigma_m^\infty$. The (constructive) dimension of X relative to π is

$$\dim^\pi(X) = \inf\{s \in [0, \infty) | X \subseteq S^\infty[d] \text{ for some lower semicomputable } \pi\text{-}s\text{-gale } d\}.$$

For a sequence $T \in \Sigma_m^\infty$ we write $\dim^\pi(T)$ for $\dim^\pi(\{T\})$.

The constructive dimension of a sequence T characterizes the information density of the sequence. In fact, an alternative Kolmogorov complexity characterization of dimension was given in [19, 1, 18]. Fix a universal prefix free TM U. For any string x, the prefix-complexity of x denoted $\mathrm{K}(x)$, is the size of the shortest binary program p such that U on input p produces x. The definition does not depend on the choice for U up to an additive constant.

Definition. Let $w \in \Sigma_m^*$. The Shannon *self-information* of w with respect to π is $\mathcal{I}_\pi(w) = \log \frac{1}{\pi(w)} = \sum_{i=0}^{|w|-1} \log \frac{1}{\pi(w[i])}$, where the logarithm is base-2 [6].

Theorem 2.1 *[19, 16, 1, 18] Let $T \in \Sigma_m^\infty$. Then*

$$\dim^\pi(T) = \liminf_{j\to\infty} \frac{\mathrm{K}(T[0..j-1])}{\mathcal{I}_\pi(T[0..j-1])}. \tag{2.1}$$

We now describe a similar dimension notion for single points in Euclidean spaces. For any $x \in \mathbb{R}$ and $r \in \mathbb{N}$, consider the Kolmogorov complexity of x at precision r given by

$$\mathrm{K}_r(x) = \min\{\mathrm{K}(q) \mid q \in \mathbb{Q} \text{ and } |x - r| \leq 2^{-r}\}.$$

The dimension of points in $\mathbb{R}^n$ is defined similarly to the Kolmogorov characterization in Theorem 2.1.

Definition. Let $x \in \mathbb{R}^n$. The *dimension* of x is

$$\dim(x) = \liminf_{r\to\infty} \frac{\mathrm{K}_r(x)}{r}. \tag{2.2}$$

2.3 Self-Similar Fractals

Many famous fractals are made up of parts that are resized copies of the whole set. As an example, the middle third Cantor set is the union of two shrunken version of the whole set. Such fractals are called self-similar.

Self-similar fractals are formally defined as the unique invariant set under a family of contracting similarities $S_0, \ldots, S_{m-1}$, where for every $i \leq m-1$ and $x, y \in \mathbb{R}^n$, $S_i : D \to D$ (where $D \subset \mathbb{R}^n$ is closed) and $|S_i(x) - S_i(y)| = c_i|x - y|$, where $c_i \in (0,1)$.

With the example of the middle third Cantor set F, letting $S_0(x) = 1/3x$ and $S_1(x) = 1/3x+2/3$ ($S_0, S_1 : \mathbb{R} \to \mathbb{R}$) the invariance property can be expressed as $F = S_0(F) \cup S_1(F)$, where $S_0(F)$ (resp. $S_1(F)$) is the resized copy of F placed on the left (resp. right).

Formally a finite sequence $S = (S_0, \ldots, S_{m-1})$ of two or more contracting similarities on a nonempty, closed set $D \subseteq \mathbb{R}^n$, is called an *iterated function system (IFS)*. We call D the *domain* of S, writing $D = \mathrm{dom}(S)$. Each S_i has a contraction ratio $c_i \in (0,1)$. Let $\mathcal{K}(D)$ be the set of compact subsets of D, and $\mathcal{K}(S) = \mathcal{K}(\mathrm{dom}(S))$. S induces a function $S : \mathcal{K}(S) \to \mathcal{K}(S)$ defined by $S(A) = \cup_{i=0}^{m-1} S_i(A)$.

Going back to the example of the middle third Cantor set, using the alphabet $\Sigma = \{0,1\}$ to refer to the contracting similarities S_0 and S_1, then each point P in F can be specified by an infinite sequence $T \in \Sigma^\infty$, that codes for the infinite sequence of similarities $S_{T[0]}, S_{T[1]}, \ldots$ that when applied successively to $A = [0,1]$, yield P. Denoting P by $S(T)$, The middle third Cantor set can be expressed as the set of points encoded by all infinite sequences $T \in \Sigma^\infty$, i.e. $F(S) = \{S(T) \mid T \in \Sigma^\infty\}$.

The general case is similar: Let $S = (S_0, \ldots, S_{m-1})$ be an IFS. Let $A \in \mathcal{K}(S)$ be such that $S(A) \subseteq A$. Consider the function $S_A : \Sigma_m^* \to \mathcal{K}(S)$ defined by the following induction. $S_A(\lambda) = A$ and $S_A(wi) = S_i(S_A(w))$ for every $w \in \Sigma_m^*$, $i \in \Sigma_m$. Because all the contraction ratios c_i are smaller that 1, it is easy to see that for each sequence $T \in \Sigma_m^\infty$ there is a unique point $S_A(T) \in \mathbb{R}^n$ such that $\cap_{w \sqsubset T} S_A(w) = \{S_A(T)\}$. We call T a *coding sequence* for the point $S_A(T)$. It is well known [10] that the function $S_A : \Sigma_m^\infty \to \mathbb{R}^n$ does not depend on the choice of A i.e., for any $A, B \in \mathcal{K}(D)$ such that $S(A) \subseteq A$ and $S(B) \subseteq B$, we have $S_A = S_B$. Thus for any IFS S, the function $S : \Sigma_m^\infty \to \mathbb{R}^n$ obtained by letting $S = S_A$ for some set $A \in \mathcal{K}(D)$ such that $S(A) \subseteq A$ is well defined. The *attractor* of the IFS S is given by $F(S) = \{S(T) | \, T \in \Sigma_m^\infty\}$.

Because the sets $S_0(D), \ldots, S_{m-1}(D)$ need not be disjoint, a point in $F(S)$ can admit more than one coding sequence T.

An attractor F of an IFS S is a *self-similar fractal* if the sets $S_0(D), \ldots, S_{m-1}(D)$ are "almost" disjoint i.e., if F satisfies the following *open set condition.*

Definition. An IFS $S = (S_0, \ldots, S_{m-1})$ with domain D satisfies the *open set condition* if there exists a nonempty, bounded, open set $G \subseteq D$ such that $S_0(G), \ldots, S_{m-1}(G)$ are disjoint subsets of G.

It is a classical result of Moran [20] and Falconer [9] that for any self similar fractal $F(S)$, the box, packing and Hausdorff dimension all coincide and are equal to the unique solution $\mathrm{sdim}(F)$ of equation $\sum_{i=0}^{m-1} c_i^{\mathrm{sdim}(F)} = 1$, where the c_i's are the compression ratios of S.

Iterated function systems induce probability measures on alphabets in the following manner.

Definition. The *similarity probability measure* of an IFS $S = (S_0, \ldots, S_{m-1})$ with contraction ratios $c_0, \ldots, c_{m-1}$ is the probability measure π_S on the alphabet Σ_m defined by

$$\pi_S(i) = c_i^{\mathrm{sdim}(S)}$$

for all $i \in \Sigma_m$.

In this paper, we are interested in computable IFSs. Here is a definition.

Definition. An IFS $S = (S_0, \ldots, S_{m-1})$ is computable if $\mathrm{dom}(S)$ is a computable set and the functions $S_0, \ldots, S_{m-1}$ are computable.

Definition. A *computably self-similar fractal* is a set $F \subseteq \mathbb{R}^n$ that is the attractor of an IFS that is computable and satisfies the open set condition.

The following theorem is our starting point.

Theorem 2.2 *[18]. If $F \subseteq \mathbb{R}^n$ is a computably self-similar fractal given by a computable IFS S, then, for all points $x \in F$ and all coding sequences U of x,*

$$\dim(x) = \mathrm{sdim}(F)\dim^{\pi_S}(U).$$

3 The Selector-Coder Game

The random fractals that we consider in this paper are randomly selected subfractals of a given computably self-similar fractal. This section explains this random selection process in terms of a two-player game.

Let n, m, and k be integers with $n \geq 1$ and $m \geq k \geq 2$. Let $F = F(S) \subseteq \mathbb{R}^n$ be a self-similar fractal given by a computable IFS $S = (S_0, \ldots, S_{m-1})$ satisfying the open set condition. Recall that

$$F = \{S(U) \mid U \in \Sigma_m^\infty\},$$

i.e., each point $x \in F$ is of the form $x = S(U)$ for some coding sequence $U \in \Sigma_m^\infty$. We are interested in certain randomly selected subfractals of the fractal F. It is easiest to specify such a subfractal by saying *which* coding sequences $U \in \Sigma_m^\infty$ give rise to points $S(U)$ in the subfractal.

Intuitively, our subfractal consists of points $S(U)$ for which U is the outcome of a game played by a *selector* and a *coder*. During each round $t = 0, 1, \ldots$ of this game, the tth symbol $U[t] \in \Sigma_m$ of U is determined by the following choices.

(i) The selector chooses a k-element subset A of Σ_m.

(ii) The coder chooses an element i of Σ_k.

The symbol $U[t]$ is then the ith element of A.

More formally, recall that $[\Sigma_m]^k$ is the set of all k-element subsets of Σ_m. Given a set $A \in [\Sigma_m]^k$ and an index $i \in \Sigma_k$, we let A_i denote the ith element of A in the standard ordering of Σ_m. Thus $A = \{A_0, \ldots, A_{k-1}\}$ and $A_0 < \cdots < A_{k-1}$.

The following definition allows the selector's choice of the set A to depend upon the coder's earlier choices of symbols in Σ_k.

Definition. An $\binom{m}{k}$-*selector* (or, when m and k are clear from the context, a *selector*) is a function

$$\sigma : \Sigma_k^* \to [\Sigma_m]^k.$$

We write $\mathrm{SEL}\binom{m}{k}$ for the set of all $\binom{m}{k}$-selectors.

For the purpose defining random subfractals, it does not really matter *how* the coder makes its choices, i.e., we can identify the coder with the sequence of choices that it makes.

Definition. A *coder* is a sequence $T \in \Sigma_k^\infty$.

Once a selector and a coder have been chosen, the outcome of the selector-coder game is determined.

Definition. Let $\sigma \in \mathrm{SEL}\binom{m}{k}$ be a selector, and let $T \in \Sigma_k^\infty$ be a coder. The *outcome* of (the selector-coder game played between) σ and T is the sequence $\sigma * T \in \Sigma_m^\infty$ defined by

$$(\sigma * T)[t] = \sigma(T[0..t-1])_{T[t]}$$

for all $t \in \mathbb{N}$.

Our intent, captured in the next definition, is for each selector σ to specify (select) the subfractal F_σ of F consisting of all points $S(U)$ for which U is an outcome of playing σ against some coder.

Definition. For each selector $\sigma \in \mathrm{SEL}\binom{m}{k}$, the *subfractal* of F *selected by* σ is the set

$$F_\sigma = \{ S(\sigma * T) \mid T \in \Sigma_k^\infty \}.$$

The following observation, which follows immediately from Theorem 2.2, reduces our study of the dimensions of points in F_σ to a study of the dimensions of sequences of the form $\sigma * T$.

Observation 3.1 *For each point $x = S(\sigma * T) \in F_\sigma$ we have*

$$\dim(x) = \mathrm{sdim}(F)\dim^{\pi_S}(\sigma * T).$$

Our interest here is in randomly selected subfractals of F, by which we mean subfractals F_σ of F for which the selector σ is random with respect to some probability measure. That is, we are interested in the case where the coder is playing a "game against nature". To make this idea precise, we identify each selector $\sigma : \Sigma_k^* \to [\Sigma_m]^k$ with its *characteristic* sequence $\chi_\sigma \in ([\Sigma_m]^k)^\infty$ defined by

$$\chi_\sigma[i] = \sigma(s_i^{(k)})$$

for all $i \in \mathbb{N}$. Note that χ_σ is an infinite sequence over the $\binom{m}{k}$-element *alphabet* $[\Sigma_m]^k$.

We now have two ways to visualize a selector σ. The original definition suggests that we regard a selector $\sigma : \Sigma_k^* \to [\Sigma_m]^k$ as a *labeled tree* in which this underlying tree is Σ_k^* (i.e., the root is λ, and each vertex w has the left-to-right children $w0$, $w1$, ..., $w(k-1)$) and each vertex w has the set $\sigma(w) \in [\Sigma_m]^k$ as its label. The identification of σ with its characteristic sequence $\chi_\sigma \in ([\Sigma_m]^k)^\infty$ suggests that we regard σ as a *breadthfirst traversal* of this labeled tree. As we shall see, these are both useful perspectives.

We often analyze selectors, coders, and their outcomes in terms of finite "initial segments". To this end, we define a string $u \in ([\Sigma_m]^k)^*$ to be a *prefix* of a selector σ, and we write $u \sqsubseteq \sigma$, if $u \sqsubseteq \chi_\sigma$.

We next define a "finite prefix version" of the outcome operation $(\sigma, T) \mapsto \sigma * T$. This takes a bit of care, because $\sigma * T$ only depends on those values $\sigma(v)$ of σ for which $v \sqsubseteq T$.

Definition. Let $u \in ([\Sigma_m]^k)^*$ and $v \in \Sigma_k^*$. Let v' be the longest prefix of v such that $j < |u|$ for every proper prefix $s_j^{(k)} \underset{\neq}{\sqsubseteq} v'$. Then the *result* of u and v is the string $u * v \in \Sigma_m^{|v'|}$ defined by

$$(u * v)[i] = u[\text{index}^{(k)}(v[0..i-1])]_{v[i]}$$

for all $0 \leq i < |v'|$, recall that $\text{index}^{(k)}(w)$ is the index of w in the standard enumeration of Σ_k^*, i.e., $s^{(k)}_{\text{index}^{(k)}(w)} = w$.

Observation 3.2 *If $\sigma \in \text{SEL}\binom{m}{k}$, $T \in \Sigma_k^\infty$, $u \sqsubseteq \sigma$, and $v \sqsubseteq T$, then $u * v$ is the longest prefix of $\sigma * T$ that is determined by u and v.*

We now define what it means for the selector σ to be random with respect to a given probability measure.

Definition. Let $\gamma \in \Delta([\Sigma_m]^k)$, i.e., let γ be a probability measure on the discrete sample space $[\Sigma_m]^k$. A selector $\sigma \in \text{SEL}\binom{m}{k}$ is *random* with respect to γ (or, more simply, γ-*random*) if its characteristic sequence χ_σ is γ-random.

This paper is concerned with the following type of fractal.

Definition. Let $\gamma \in \Delta([\Sigma_m]^k)$. A γ-*random subfractal* of F is a set F_σ, where σ is a γ-random selector.

The following well-known Kolmogorov complexity characterization of γ-randomness [15] is useful.

Theorem 3.3 *A selector $\sigma \in \text{SEL}\binom{m}{k}$ is random with respect to a computable probability measure $\gamma \in \Delta([\Sigma_m]^k)$ if and only if every sufficiently long prefix $u \sqsubseteq \sigma$ satisfies*

$$\text{K}(u) > \mathcal{I}_\gamma(u) \log \binom{m}{k}.$$

4 Similarity-random subfractals

This is the main section of the paper. We investigate the dimension spectra of a natural class of random subfractals of a self-similar fractal. This class is somewhat restrictive, but it exhibits several subtleties of the interactions between randomness and dimension.

As before, let n, m, and k be integers with $n \geq 1$ and $m \geq k \geq 2$. Let $F = F(S) \subseteq \mathbb{R}^n$ be a computably self-similar fractal given by a computable IFS $S = (S_0, \ldots, S_{m-1})$ satisfying the open set condition.

Definition. The *similarity probability measure* induced by S (equivalently, by F) on $[\Sigma_m]^k$ is the probability measure $\hat{\pi_s} \in \Delta([\Sigma_m]^k)$ given by

$$\hat{\pi_s}(A) = \frac{\pi_S(A)}{\binom{m-1}{k-1}}$$

for each $A \in [\Sigma_m]^k$. Here π_s is the similarity probability measure on Σ_m defined in section 2, and we write $\pi_S(A) = \sum_{i \in A} \pi_S(i)$.

It is routine to verify that $\sum_{A \in [\Sigma_m]^k} \hat{\pi_s}(A) = 1$, whence $\hat{\pi_s} \in \Delta([\Sigma_m]^k)$. It should also be noted that, if the similarities $S_0, \ldots, S_{m-1}$ all have the same contraction ratio, then π_s is the uniform probability measure on Σ_m, and $\hat{\pi_s}$ is the uniform probability measure on $[\Sigma_m]^k$.

Definition.

1. A selector $\sigma \in \mathrm{SEL}\binom{m}{k}$ is *similarity random* if it is $\hat{\pi_s}$-random.
2. A *similarity random subfractal* of F is a subfractal F_σ of F (as defined in section 3), where σ is a similarity random selector.

Our objective is to prove the following.

Theorem 4.1 (main theorem) *For every similarity random subfractal F_σ of F, the dimension spectrum $\mathrm{sp}(F_\sigma)$ is an interval satisfying*

$$[s^* \frac{\log m - \log k}{\log \frac{1}{a}}, s^*] \subseteq \mathrm{sp}(F_\sigma) \subseteq [s^* \frac{\log m - \log k}{\log \frac{1}{A}}, s^*],$$

where $s^* = \mathrm{sdim}(F)$, $a = \min\{\pi_S(i) \mid i \in \Sigma_m\}$, and $A = \max\{\pi_S(i) \mid i \in \Sigma_m\}$. In particular, if all the contraction ratios of F have the same value c, then every similarity-random (i.e., uniformly random) subfractal F_σ of F has dimension spectrum

$$\mathrm{sp}(F_\sigma) = [s^*(1 - \frac{\log k}{\log m}), s^*],$$

where $s^* = \mathrm{sdim}(F) = (\log m)/(\log \frac{1}{c})$.

Example 4.2 *Let F be the standard Sierpinski triangle. This is given by and IFS $S = (S_0, S_1, S_2)$ in which all three contraction ratios are $c = \frac{1}{2}$, so $\dim_{\mathrm{H}}(F) = \mathrm{sdim}(F) = \log 3$. If σ is a uniformly random selector that chooses two of the contractions S_0, S_1, S_2 at every stage, then Theorem 4.1 says that the resulting random subfractal F_σ of F has dimension spectrum*

$$\mathrm{sp}(F_\sigma) = [(\log 3) - 1, \log 3] \approx [0.585, 1.585].$$

We now turn to the proof of Theorem 4.1. Let J be the set of all possible values of $\dim^{pis}(\sigma * T)$. By Observation 3.1 it suffices to prove the following three things.

- J is an interval. (4.1)

- $1 \in J$ (4.2)
- $\frac{\log m - \log k}{\log \frac{1}{a}} \leq \inf J \leq \frac{\log m - \log k}{\log \frac{1}{A}}$ (4.3)

It is routine, though delicate, to use a "back-and-forth" construction to show that J is convex, whence (4.1) holds. We omit the details here and focus on the more interesting components of the proof.

The following lemma and theorem establish that (4.2) holds. Proofs appear in the appendix.

Lemma 4.3 *There is a distribution $\gamma \in \Delta(\Sigma_k)$ such that the outcome operation $(\sigma, T) \mapsto \sigma * T$ is measure-preserving when using distributions $\pi_S \in \Delta(\Sigma_m)$ and $\hat{\pi_s} \in \Delta([\Sigma_m]^k)$.*

Theorem 4.4 *Let γ be the distribution given by the previous lemma. If a coder T is γ-random relative to a similarity-random selector σ, then the coding sequence $\sigma * T$ is π_S-random, so $\dim^{\pi_S}(\sigma * T) = 1$.*

The proof of Theorem 4.4 uses Lemma 4.3 and van Lambalgen's theorem [22].

We now turn to (4.3), which concerns the left endpoint of the interval J. The question is now how small the coder T can force the dimension $\dim^{\pi_S}(\sigma * T)$ to be. More intuitively, how much of the randomness in σ can the coder "cancel"? The following theorem places an upper bound on the amount of such cancellation and thereby establishes the left-hand inequality in (4.3).

Theorem 4.5 *If σ is a similarity-random selector, then for every coder T,*

$$\dim^{\pi_S}(\sigma * T) \geq \frac{\log m - \log k}{-\log a},$$

where $a = \min\{\pi_S(i) \mid i \in \Sigma_m\}$.

The proof is a Kolmogorov complexity argument. Roughly, if a prefix w of $\sigma * T$ can be compressed to $(\frac{\log m - \log k}{-\log a} - \epsilon)\mathcal{I}_\pi(w)$ bits, then σ is "somewhat" compressible, hence nonrandom by Theorem 3.3. Details appear in the appendix.

To prove the right-hand inequality in (4.3) we need a strategy by which T can cancel as much as the randomness in σ as possible. A tempting strategy for this is $T = 0^\infty$, which always chooses the minimum element of the set chosen by σ. Consider this coder T in the following specific context.

Example 4.6 *Let F and σ be as in Example 4.2, and let $T = 0^\infty$. It is easy to see that the outcome $\sigma * T$ is α-random, where $\alpha \in \Delta(\Sigma_3)$ is given by $\alpha(0) = \frac{2}{3}$, $\alpha(1) = \frac{1}{3}$, and $\alpha(2) = 0$. It follows by Theorem 7.7 of [16] that $\dim(\sigma * T) = \mathcal{H}_3(\alpha) = 1 - \frac{2}{3 \log 3} \approx 0.58$, whence by Observation 3.1 that $\dim(S(\sigma * T)) = (\log 3) - \frac{2}{3} \approx 0.918$.*

This example shows that the coder $T = 0^\infty$ does indeed cancel some of the randomness in σ, but not enough to reach the left endpoint of the spectrum claimed in Example 4.2. The following theorem uses a nonconstructive strategy to establish the right-hand inequality of (4.3). The proof is in the appendix.

Theorem 4.7 *For every similarity-random selector σ, there is a coder T such that*

$$\dim^{\pi_S}(\sigma * T) \leq \frac{\log m - \log k}{-\log A},$$

where $A = \max\{\pi_S(i) \mid i \in \Sigma_m\}$.

This concludes the proof of our main theorem.

5 More General Random Subfractals

5.1 Upper Bound For Product Distribution

In this section, we investigate the upper bound for points in random subfractals generated from product measures on $\mathrm{SEL}\binom{m}{k}$, i.e., measures on $\mathrm{SEL}\binom{m}{k}$ that are products of measures in $\Delta([\Sigma_m]^k)$.

The well-known max-flow min-cut theorem is particularly useful in our investigation, so we include a definition of the flow network for completeness.

Definition. A *flow network* is a tuple $N := \langle G, s, t, c \rangle$, where $G = (V, E)$ is a directed graph, $s, t \in V$, and $c : E \to \mathbb{R}$ is a *capacity function.* A *flow* in N is a function $f : \mathrm{E} \to \mathbb{R}$ that satisfies the following conditions:

1. $\sum_{(u,v)\in E} f((u,v)) = \sum_{(v,u)\in E} f((v,u))$ for all $v \in V - \{s,t\}$.
2. $f(e) \le c(e)$ for all $e \in E$.

In the following, we give establish conditions under which the dimensions of points in random subfractals can achieve the dimension of the fractal from which the random subfractal is generated. Our proof makes essential use of the max-flow min-cut theorem.

Theorem 5.1.1 *Let $\rho \in \Delta(\Sigma_m)$ and $\gamma \in \Delta([\Sigma_m]^k)$ be such that the following condition holds*

$$\sum_{A\cap C\neq\varnothing} \gamma(A) \ge \sum_{i\in C} \rho(i),\ \forall C \subseteq \Sigma_m. \tag{5.1.1}$$

Then for every γ-random selector σ there is a $T \in \Sigma_k^\infty$ such that $\sigma * T$ is ρ-random.

The flow network construction in the proof of Theorem 5.1.1 not only tells us when the points in a random subfractal can achieve the dimension of the original fractal, but also provides us a way to find out the maximum dimension of points in a random subfractal when the dimension of the original fractal is not achievable.

Theorem 5.1.2 *Let $\rho \in \Delta(\Sigma_m)$ and $\gamma \in \Delta([\Sigma_m]^k)$ be such that condition* (5.1.1) *holds. Then for every γ-random selector σ there is a $T \in \Sigma_k^\infty$ such that*
$\dim(S(\sigma * T)) = \mathrm{cdim}(F_\sigma(S)) = \dim_{\mathrm{H}}(F(S)) \dim^{\pi_S}(\sigma * T)$, *with*

$$\dim^{\pi_S}(\sigma * T) = \frac{\mathcal{H}(\rho)}{\mathcal{H}(\rho) + D(\rho||\pi_S)},$$

which is the π_S-dimension of a ρ-random sequence.

Corollary 5.1.3 *Let $\gamma \in \Delta([\Sigma_m]^k)$ be such the following condition holds*

$$\sum_{A\cap C\neq\varnothing} \gamma(A) \ge \sum_{i\in C} \pi_S(i),\ \forall C \subseteq \Sigma_m. \tag{5.1.2}$$

Then for every γ-random selector σ there is a $T \in \Sigma_k^\infty$ such that $\sigma * T$ is π_S-random, and

$$\dim(S(\sigma * T)) = \mathrm{cdim}(F_\sigma(S)) = \dim_{\mathrm{H}}(F(S)).$$

Corollary 5.1.4 *Let $\gamma \in \Delta([\Sigma_m]^k)$. Then for every γ-random selector σ there is a $T \in \Sigma_k^\infty$ such that* $\dim(S(\sigma * T)) = \mathrm{cdim}(F_\sigma(S)) = \dim_{\mathrm{H}}(F(S))A$*, where*

$$A = \max\{E_\rho \log \pi_S \mid \rho \textit{ satisfies condition } (5.1.1)\}$$

Note. Maximizing $E_\rho \log \pi_S$ is equivalent to minimizing $\mathcal{H}(\rho) + D(\rho||\pi_S)$, where

$$D(\rho||\pi_S) = E_\rho \log \frac{\rho}{\pi_S}$$

is the Kullback-Leibler divergence between two probability measures.

Remark. Condition (5.1.1) is equivalent to

$$\sum_{A \in \mathcal{C}} \gamma(A) \leq \sum_{i \in \cup_{A\in\mathcal{C}} A} \rho(i),\ \forall \mathcal{C} \subseteq [\Sigma_m]^k. \tag{5.1.3}$$

This is because both γ and ρ are probability measures, i.e., both sum to 1 and the problem is symmetric. Also note that the achievability of π_S (that is, the existence of $T \in \Sigma_k^\infty$ such that $\sigma * T$ is π_S-random) also implies condition (5.1.2).

5.2 Lower Bound For General Computable Distribution

In the following theorem, we provide a dimension lower bound tool for more general computable probability measures on $\mathrm{SEL}\binom{m}{k}$.

Theorem 5.2.1 *Let $\gamma \in \Delta(\mathrm{SEL}\binom{m}{k})$ be computable. Let σ be an γ-random selector.*

For each $A \in [\Sigma_m]^k$, $U \in ([\Sigma_m]^k)^*$, $i \in \Sigma_m$, and $w \in \Sigma_m^*$ define $\rho(A|U,i)$ and $\rho_w \in \Delta(\mathrm{SEL}\binom{m}{k})$ as follows.

$$\rho(A|U,i) = \frac{\gamma(A|U)}{\sum_{B\in[\Sigma_m]^k, i\in B} \gamma(B|U)},$$

$$\rho_w(UA) = \begin{cases} 1 & U = A = \lambda, \\ \rho_w(U)\rho(A|U, w[|T|-1]) & (\exists T \in \Sigma_k^{\mathrm{depth}(UA)}) UA * T \sqsubseteq w, |U * T| = |UA * T| - 1, \\ \rho_w(U)\gamma(A|U) & (\exists T \in \Sigma_k^{\mathrm{depth}(UA)}) UA * T = U * T \sqsubseteq w, \\ 0 & \textit{otherwise.} \end{cases}$$

Fix $T \in \Sigma_k^\infty$ and let $x = \sigma * T$. Then

$$\dim(x) \geq \liminf_{n\to\infty} \frac{\log\left(\frac{\prod_{i\in\mathcal{U}_{x[0..n-1]}} \rho_{x[0..n-1]}(\sigma[i]|\sigma[0..i-1])}{\prod_{i\in\mathcal{U}_{x[0..n-1]}} \gamma(\sigma[i]|\sigma[0..i-1])}\right)}{n \log m},$$

where $\mathcal{U}_w = \left\{\mathrm{index}^{(k)}(w') \mid w \neq w' \sqsubseteq w\right\}$ for all $w \in \Sigma_m^*$.

Remark. In this theorem, the bound only depends on γ. It is easy to verify by substituting the correct probability measure that Theorem 4.5 can be derived from Theorem 5.2.1.

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Optional Technical Appendix

Proof (of Lemma 4.3). We can explicitly define $\gamma(0)$, then $\gamma(1)$, etc using equations

$$\pi_S(0) = \hat{\pi}_s(0)\gamma(0)$$

$$\pi_S(1) = \sum_{1\in A, 0\notin A} \hat{\pi}_s(A)\gamma(0) + \sum_{0,1\in A} \hat{\pi}_s(A)\gamma(1)$$

...

□

Proof (of Theorem 4.4). The proof of this result uses van Lambalgen's theorem [22] and the fact that the outcome operation $(\sigma, T) \mapsto \sigma * T$ is measure-preserving.

We prove this theorem by proving that if σ is $\hat{\pi}_s$-random and $\sigma * T$ is not π_S-random, then T is not γ-random relative to σ.

Let $d : \Sigma_m^\infty \to [0, \infty]$ be an optimal constructive π_S-martingale on Σ_m^∞. Without loss of generality, assume that $d(\lambda) < 1$.

For each $n \in \mathbb{N}$, let

$$Z_n = \left\{ U \in \Sigma_m^\infty \,\middle|\, (\exists u \sqsubseteq U) d(u) > 2^{2n} \right\}.$$

It is clear that $\{Z_n\}_{n\in\mathbb{N}}$ is uniformly computably enumerable (or Σ_1^0) and $\pi_S(Z_n) < 2^{-2n}$.

For each $\sigma' \in \mathrm{SEL}\binom{m}{k}$ and each $n \in \mathbb{N}$, let

$$Y_n^{\sigma'} = \left\{ T' \in \Sigma_k^\infty \,\middle|\, (\exists u \sqsubseteq \sigma' * T') d(u) > 2^{2n} \right\}.$$

Let

$$X_n = \left\{ \sigma' \in \mathrm{SEL}\binom{m}{k} \,\middle|\, \gamma(Y_n^{\sigma'}) > 2^{-n} \right\}.$$

It is clear that $\{X_n\}_{n\in\mathbb{N}}$ is Σ_1^0. Then since $*$ is measure preserving, $\pi_S(Z_n) \geq \hat{\pi}_s(X_n) \cdot \gamma(Y_n^{\sigma'})$ and

$$\hat{\pi}_s(X_n) \cdot 2^{-n} \leq \pi_S(Z_n) < 2^{-2n}.$$

Therefore, $\hat{\pi}_s(X_n) < 2^{-n}$.

Since σ is $\hat{\pi}_s$-random, $|\{n \in \mathbb{N} \,|\, \sigma \in X_n\}| < \infty$ and thus there exists $n_0 \in \mathbb{N}$ such that for all $n_0 \leq n \in \mathbb{N}$,

$$\gamma(Y_n^\sigma) \leq 2^{-n}.$$

Note that $\{Y_n^\sigma\}_{n\in\mathbb{N}}$ is uniformly Σ_1^0 in σ. Since $\sigma * T$ is not π_S-random, $T \in \bigcap_{n\in\mathbb{N}} Y_n^\sigma$, which is a σ-effective measure 0 set. Therefore T is not γ-random relative to σ. □

Lemma .0.2 (Technical Lemma) *There is a constant $c \in \mathbb{N}$ with the following property. For every selector σ, every coder T, and every prefix $w \sqsubseteq \sigma * T$, if u_w is the $(k^{|w|} - 1)/(k-1)$-symbol prefix of σ (i.e., the prefix of σ that determines all labels at depth less than $|w|$ when σ is viewed as a labeled tree), then*

$$\mathrm{K}(u_w) \leq |u_w| \log \binom{m}{k} + \mathrm{K}(w) - |w| \log \tfrac{m}{k} + c. \tag{.0.1}$$

Proof. Let M be a self-delimiting Turing machine that does the following on program $\pi \in \{0,1\}^*$. In order for the computation of M to succeed, π must be of the form

$$\pi = \pi_0 \pi_1 z',$$

where the strings π_0, π_1, $z' \in \{0,1\}^*$ have the following properties.

1. $U(\pi_0)$ is (a canonical binary encoding of) a string $w \in \Sigma_m^*$.

2. $U(\pi_1, w)$ is (a canonical binary encoding of) a string $y \in ([\Sigma_m]^{k-1})^*$ satisfying $|y| = |w|$ and $w[i] \notin y[i]$ for all $0 \leq i < |w|$.

3. z' is (a canonical binary encoding of) a string $z \in ([\Sigma_m]^k)^*$ satisfying

$$|z| = \frac{k^{|w|} - 1}{k - 1} - |w|.$$

If the above conditions hold, then $M(\pi)$ executes the algorithm in Figure 1.

begin
$u, v := \lambda, \lambda$;
for $i = 0$ **to** $(k^{|w|} - 1)/(k - 1) - 1$ **do**
if $s_i^{(k)} = v$
then begin
    $A := \{\mathrm{head}(w)\} \cup \mathrm{head}(y)$;
    $u := uA$;
    $v := vj$, where $A_j = \mathrm{head}(w)$;
    $w, y := \mathrm{tail}(w), \mathrm{tail}(y)$
    end
else begin
    $u := u\mathrm{head}(z)$;
    $z := \mathrm{tail}(z)$
    end;
output u
end

Figure 1: Algorithm for M in the proof of Lemma .0.2

Let $c_M \in \mathbb{N}$ be an optimality constant for M, so that

$$\mathrm{K}(u) \leq \mathrm{K}_M(u) + c_M \tag{.0.2}$$

holds for all $u \in ([\Sigma_m]^k)^*$. By standard techniques, there is a constant $c_1 \in \mathbb{N}$ such that

$$\mathrm{K}(y|w) \leq |w| \log \binom{m-1}{k-1} + c_1 \tag{.0.3}$$

holds for all strings $w \in \Sigma_m^*$ and $y \in ([\Sigma_m]^{k-1})^*$ satisfying $|y| = |w|$ and $w[i] \notin y[i]$ for all $0 \leq i < |w|$. Also by standard techniques, there is a constant $c_2 \in \mathbb{N}$ such that each string $z \in ([\Sigma_m]^k)^*$ has a canonical binary encoding $z' \in \{0,1\}^*$ satisfying

$$|z'| = |z| \log \binom{m}{k} + c_2. \tag{.0.4}$$

Let

$$c + c_M + c_1 + c_2. \tag{.0.5}$$

To see that c has the desired property, let σ, T, w, and u_w be as given. Let v be the prefix of T with $|v| = |w|$. Define $y \in ([\Sigma_m]^{k-1})^{|w|}$ by

$$y[i] = \sigma(v[0..i-1]) - \{w[i]\}$$

for all $0 \le i < |w|$. Let z be the string obtained from u_w by deleting all the symbols $u_w[\mathrm{index}^{(k)}(v')]$ for which $\lambda \ne v' \sqsubseteq v$. (Note that $u_w[\mathrm{index}^{(k)}(v')] = \sigma(v')$.). Then the algorithm of Figure 1 "reconstructs" the strings $u = u_w$ and v from the strings w, y, and z. It follows that

$$\mathrm{K}_M(u_w) \le \mathrm{K}(w) + \mathrm{K}(y|w) + |z'|,$$

where z' is the canonical binary encoding of z. By (.0.2), (.0.3), and (.0.4), we then have

$$\begin{aligned}\mathrm{K}(u_w) &\le \mathrm{K}(w) + |w| \log \tbinom{m-1}{k-1} + |z| \log \tbinom{m}{k} + c \\ &= \mathrm{K}(w) + |w| \log \tbinom{m-1}{k-1} + (|u_w| - |w|) \log \tbinom{m}{k} + c \\ &= |u_w| \log \tbinom{m}{k} + \mathrm{K}(w) - |w| \log \tfrac{m}{k} + c,\end{aligned}$$

i.e., (.0.1) holds. □

Proof (of Theorem 4.5).

Let σ and T be as given. Choose $c_\sigma \in \mathbb{N}$ as in Theorem 3.3, and choose $c \in \mathbb{N}$ as in Lemma .0.2. Then, for every prefix $w \sqsubseteq \sigma * T$, we have

$$\begin{aligned}|u_w| \log \tbinom{m}{k} - c_\sigma &\le \mathrm{K}(u_w) \\ &\le |u_w| \log \tbinom{m}{k} + \mathrm{K}(w) - |w| \log \tfrac{m}{k} + c,\end{aligned}$$

so

$$\begin{aligned}\mathrm{K}(w) &\ge |w| \log \tfrac{m}{k} - c_\sigma - c \\ &= |w|(\log \tfrac{m}{k} - o(1))\end{aligned}$$

as $w \mapsto \sigma * T$, so

$$\begin{aligned}\dim^{\pi_S}(\sigma * T) &= \liminf_{w \to \sigma * T} \frac{\mathrm{K}(w)}{\mathcal{I}_{\pi_S}(w)} \\ &\ge \frac{\log \frac{m}{k}}{-\log a_S}.\end{aligned}$$

□

Proof (of Theorem 4.7). Let $\alpha > 1 - \log k / \log m$. Let $u \in \Sigma_m^*$, $n \in \mathbb{N}$. We define the set

$$Z_n^u = \{\sigma \mid \forall T \text{ with } u \sqsubseteq T, \mathrm{K}(\sigma * T[0..n-1] \ge \alpha n \log m\}.$$

Let $i \in \Sigma_m$ be such that $\pi_s(i) \ge \frac{1}{m}$. Then $\hat{\pi_s}(i) \ge \frac{k}{m}$.

If $\sigma \in Z_n^u$, for instance for all T extending u, $\sigma * T[0..n-1] \notin \Sigma_m^{\alpha n}\{i\}^{(1-\alpha)n/}$ and therefore

$$\begin{aligned}\hat{\pi_s}(Z_n) \le \quad & \left(1 - (\hat{\pi_s}(i))^{(1-\alpha)n}\right)^{k^{\alpha n - |u|}} \\ \le \quad & \left(1 - (\frac{k}{m})^{(1-\alpha)n}\right)^{k^{\alpha n - |u|}} \\ \approx \quad & e^{-k^n / m^{(1-alpha)n}} \to_n 0.\end{aligned}$$

So $\{Z_n^u\}$ is a $\hat{\pi_s}$-Martin-Löf test for each u, and if σ is $\hat{\pi_s}$-random $\exists^\infty n \sigma \notin Z_n^u$. Therefore there is a $T \in \Sigma_k^\infty$ such that $\mathrm{K}(\sigma * T[0..n-1] < \alpha n \log m$ for infinitely many n, and $\dim^{\pi_S}(\sigma * T) \le \alpha \cdot \frac{\log m}{-\log A_S}$.

Instead of a single α we can take a decreasing rational sequence $\alpha_r \to_r 1 - \log k / \log m$ and prove that there is a $T \in \Sigma_k^\infty$ with $\dim^{\pi_S}(\sigma * T) \leq \frac{\log m - logk}{\log A_S}$. $\square$

Proof (of Theorem 5.1.1). We prove this theorem by proving that there is a probability measure that depends on σ, according to which, a random T has the desired property. We first formulate this problem in terms of a network flow problem. We then prove that condition (5.1.1) imply the maximum flow in our flow network is exactly 1 and construct the desired probability measure based on a maximum flow.

Let $G = (V, E)$ be a directed graph such that

$$V = \{s, t\} \cup [\Sigma_m]^k \cup \Sigma_m$$

and

$$E = \left\{ s, A \,\middle|\, A \in [\Sigma_m]^k \right\} \cup \{ i, t \mid i \in \Sigma_m \} \cup \{ A, i \mid i \in A \}.$$

Let

$$c(e) = \begin{cases} \gamma(A) & \text{if } e = s, A \\ \infty & \text{if } e = A, i \\ \rho(i) & \text{if } e = i, t. \end{cases}$$

be a capacity function. Then $N = \langle G, s, t, c \rangle$ is a flow network.

Since γ and ρ are probability measures, it is clear that for any flow $f : E \to \mathbb{R}$ in N, $f(s,t) \leq 1$. It is also clear that the smallest cut in N has capacity less than or equal to 1.

By the min-cut/max-flow theorem, it suffices to show that the minimum cut of G has capacity at least 1.

Note that for any cut that contains an edge in $\{A, i \mid i \in A\}$, the capacity of the cut is ∞. Any such cut cannot be a minimum cut. Let $B \cup C^c$ be a non-trivial cut of G, where $B \subseteq \left\{ s, A \,\middle|\, A \in [\Sigma_m]^k \right\}$ and $C^c \subseteq \{ i, t \mid i \in \Sigma_m \}$. (We insist here that $C^c \cup C = \{ i, t \mid i \in \Sigma_m \}$) Let $B' = \left\{ s, A \,\middle|\, A \in [\Sigma_m]^k, (A \times \{t\}) \cap C \neq \varnothing \right\}$. Note that since $B \cup C^c$ is a cut, $B' \subseteq B$ and it is easy to see that $B' \cup C^c$ is a cut. So $B' \cup C^c$ has capacity at most that of $B \cup C^c$. The capacity of $B' \cup C^c$ is

$$\begin{aligned} \sum_{(s,A) \in B'} c(s, A) + \sum_{(i,t) \in C^c} c(i, t) &= \sum_{(s,A) \in B'} c(s, A) + 1 - \sum_{(i,t) \in C} c(i, t) \\ &= \sum_{A \cap C \neq \varnothing} \gamma(A) + 1 - \sum_{i \in C} \rho(i) \\ &\geq 1 \qquad \text{by condition (5.1.1).} \end{aligned}$$

Therefore, the capacity of the cut $B \cup C^c$ is at least 1. Since $B \cup C^c$ is an arbitrary non-trivial cut, the minimum cut capacity is 1 and the maximum flow is at least 1.

Let f be a flow of value 1 for the network N. For each $A \in [\Sigma_m]^k$, define the probability measure $\nu_A \in \Delta(\Sigma_k)$ by

$$\nu_A(j) = \frac{f(A, A_j)}{\gamma(A)},$$

for each $j \in \Sigma_k$. (A_j is the jth element of the set A in numerical order.)

For each $\sigma' \in \mathrm{SEL}\binom{m}{k}$, define the probability measure $\nu_{\sigma'} \in \Delta(\Sigma_k^\infty)$ by the following recursion.

$$\nu_{\sigma'}(\lambda) = 1$$

$$\nu_{\sigma'}(wa) = \nu_{\sigma'}(w)\nu_{\sigma'(w)}(a)$$

for all $w \in \Sigma_k^*$ and all $a \in \Sigma_k$.

In the following, we show that for an algorithmic ν_σ-random T, $\sigma * T$ is ρ-random. We do so by showing that if $\sigma * T$ is not ρ-random, then T is not ν_σ-random relative to σ.

Let $d : \Sigma_m^\infty \to \mathbb{R}$ be an optimal constructive ρ-martingale on Σ_m^∞ with $d(\lambda) < 1$.

For each $n \in \mathbb{N}$, let

$$Z_n = \left\{ U \in \Sigma_m^\infty \,\middle|\, (\exists u \sqsubseteq U) d(u) > 2^{2n} \right\}.$$

It is clear that $\{Z_n\}_{n\in\mathbb{N}}$ is uniformly Σ_1^0 and $\rho(Z_n) < 2^{-2n}$.

For each $\sigma' \in \mathrm{SEL}\binom{m}{k}$ and each $n \in \mathbb{N}$, let

$$Y_n^{\sigma'} = \left\{ T' \in \Sigma_k^\infty \,\middle|\, (\exists u \sqsubseteq \sigma' * T') d(u) > 2^{2n} \right\}.$$

Let

$$X_n = \left\{ \sigma' \in \mathrm{SEL}\tbinom{m}{k} \,\middle|\, \nu_{\sigma'}(Y_n^{\sigma'}) > 2^{-n} \right\}.$$

It is clear that $\{X_n\}_{n\in\mathbb{N}}$ is uniformly Σ_1^0 and $\gamma(X_n) \le 2^{-n}$. Since the joint distribution of $\gamma(\sigma')$ and $\nu_{\sigma'}$ is ρ,

$$\gamma(X_n) \cdot 2^{-n} \le \rho(Z_n) < 2^{-2n}.$$

Therefore, $\gamma(X_n) < 2^{-n}$.

Since σ is γ-random, $|\{n \in \mathbb{N} \mid \sigma \in X_n\}| < \infty$ and thus there exists $n_0 \in \mathbb{N}$ such that for all $n_0 \le n \in \mathbb{N}$,

$$\nu_\sigma(Y_n^\sigma) \le 2^{-n}.$$

Note that $\{Y_n^\sigma\}_{n\in\mathbb{N}}$ is uniformly Σ_1^0 in σ. Since $\sigma * T$ is not ρ-random, $T \in \bigcap_{n\in\mathbb{N}} Y_n^\sigma$, which is a σ-effective ν_σ measure 0 set. Therefore T is not ν_σ-random relative to σ. □

Proof (of Theorem 5.1.2). The result follows from Theorem 5.1.1 together with the Kullback-Leibler divergence Lemma. □

Proof (of Theorem 5.2.1).

In here and hereafter, $\mathrm{depth}(U) = 1 + \max\left\{ |s_i^{(k)}| \,\middle|\, i < |U| \right\}$, is intuitive the depth (the number of layers of nodes) in the tree defined by U, as U is a prefix of some selector, which we regard as a labeled tree.

Note that when $|T| \ge \mathrm{depth}(U)$, T specifies a branch of U to as far as U allows and it is possible that only a proper prefix of T is used.

When $UA * T = U * T$ for some $T \in \Sigma_k^{\mathrm{depth}(UA)}$ and $U * T \sqsubseteq w$, then the last bit A in UA is not in obvious way related to w and therefore the knowledge of w in no obvious way helps predicting A given U.

When $UA * T \sqsubseteq w$ and $|U * T| = |w| - 1$, a bit of w is partially determined by the last bit A in UA and hence the knowledge of w obviously gives some information on what A should be given U.

Define the following subprobability measure.

Let

$$\rho(U) = \sum_{w \in \Sigma_m^*} 2^{-\mathrm{K}(w)} \rho_w(U).$$

Since γ is computable and the inverse of the $*$ operator is computable, ρ is constructive and is dominated by the optimal constructive subprobability supermeasure.

Let

$$d(U) = \frac{\rho(U)}{\gamma(U)}.$$

Then d is a constructive r-supermartingale.

Since σ is γ-random,

$$\limsup_{n\to\infty} d(\sigma[0..n-1]) < \infty.$$

Let $w \in \Sigma_m^*$. Let $U \in ([\Sigma_m]^k)^*$ such that there exists some $T' \in \Sigma_k$ such that $U * T' = w$ and $\mathrm{depth}(U) = |w|$ and for any A' $\mathrm{depth}(UA') = |w| + 1$.

Note that $|\mathcal{U}_w| = \mathrm{depth}(U) = |w|$ and that $\mathcal{U}_w$ is intuitive the positions along U where the digits of w directly depend on. Let $\mathcal{U}_w^c = \{j < |U| \mid j \notin \mathcal{U}\}$. Then

$$\begin{aligned}
d(U) &= \frac{\rho(U)}{\gamma(U)} \\
&= \frac{\sum_{w'\in\Sigma_m^*} 2^{-\mathrm{K}(w')}\rho_{w'}(U)}{\gamma(U)} \\
&\geq 2^{-\mathrm{K}(w)} \frac{\rho_w(U)}{\gamma(U)} \\
&= 2^{-\mathrm{K}(w)} \frac{\prod_{i=0}^{|U|-1} \rho_w(U[i]|U[0..i-1])}{\prod_{i=0}^{|U|-1} \gamma(U[i]|U[0..i-1])} \\
&= 2^{-\mathrm{K}(w)} \frac{\prod_{i\in\mathcal{U}_w} \rho_w(U[i]|U[0..i-1])}{\prod_{i\in\mathcal{U}_w} \gamma(U[i]|U[0..i-1])} \cdot \frac{\prod_{i\in\mathcal{U}_w^c} \rho_w(U[i]|U[0..i-1])}{\prod_{i\in\mathcal{U}_w^c} \gamma(U[i]|U[0..i-1])} \\
&= 2^{-\mathrm{K}(w)} \frac{\prod_{i\in\mathcal{U}_w} \rho_w(U[i]|U[0..i-1])}{\prod_{i\in\mathcal{U}_w} \gamma(U[i]|U[0..i-1])} \cdot 1.
\end{aligned}$$

Since σ is γ-random, for every $U \sqsubseteq \sigma$

$$d(U) \leq \mathcal{O}(1).$$

Therefore,

$$2^{-\mathrm{K}(w)} \frac{\prod_{i\in\mathcal{U}_w} \rho_w(U[i]|U[0..i-1])}{\prod_{i\in\mathcal{U}_w} \gamma(U[i]|U[0..i-1])} \leq 1,$$

i.e.,

$$2^{\mathrm{K}(w)} \geq \frac{\prod_{i\in\mathcal{U}_w} \rho_w(U[i]|U[0..i-1])}{\prod_{i\in\mathcal{U}_w} \gamma(U[i]|U[0..i-1])},$$

and

$$\mathrm{K}(w) \geq \log\left(\frac{\prod_{i\in\mathcal{U}_w} \rho_w(U[i]|U[0..i-1])}{\prod_{i\in\mathcal{U}_w} \gamma(U[i]|U[0..i-1])}\right).$$

Let $x \in \sigma * \Sigma_k^\infty$ be a sequence. So the constructive dimension of x is

$$\begin{aligned}
\dim(x) &= \liminf_{n\to\infty} \frac{\mathrm{K}(x[0..n-1])}{n\log m} \\
&\geq \liminf_{n\to\infty} \frac{\log\left(\frac{\prod_{i\in\mathcal{U}_{x[0..n-1]}} \rho_{x[0..n-1]}(\sigma[i]|\sigma[0..i-1])}{\prod_{i\in\mathcal{U}_{x[0..n-1]}} \gamma(\sigma[i]|\sigma[0..i-1])}\right)}{n\log m}.
\end{aligned}$$

□