Connectivity Properties of Dimension Level Sets

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Abstract

This paper initiates the study of sets in Euclidean space $\mathbb{R}^n$ ($n \geq 2$) that are defined in terms of the dimensions of their elements. Specifically, given an interval $I \subseteq [0,1]$, we are interested in the connectivity properties of the set DIM^I consisting of all points in $\mathbb{R}^n$ whose (constructive Hausdorff) dimensions lie in the interval I. It is easy to see that the sets $\mathrm{DIM}^{[0,1)}$ and $\mathrm{DIM}^{(n-1,n]}$ are totally disconnected. In contrast, we show that the sets $\mathrm{DIM}^{[0,1]}$ and $\mathrm{DIM}^{[n-1,n]}$ are path-connected. Our proof of this fact uses geometric properties of Kolmogorov complexity in Euclidean space.

1 Introduction

Constructive dimension, an effectivization of classical Hausdorff dimension introduced in 2000 [10, 11], assigns a dimension $\dim(S) \in [0,1]$ to each sequence $S \in \mathbf{C}$, where $\mathbf{C} = \{0,1\}^\infty$ is the Cantor space. The properties of constructive dimension and its relationships with algorithmic randomness, Kolmogorov complexity, and other topics in the theory of computing have been extensively investigated over the past few years [6]. Intuitively, the dimension of a sequence S is the asymptotic density of information in S [13, 12].

Constructive dimension on the Cantor space naturally induces constructive dimensions on Euclidean spaces. Specifically, for each positive integer n, constructive dimension assigns a *dimension* $\dim(x) \in [0,n]$ to each *individual* point $x \in \mathbb{R}^n$. For each real number $\alpha \in [0,n]$, there do in fact exist points $x \in \mathbb{R}^n$ with $\dim(x) = \alpha$ [11]. Although it may at first seem counter-intuitive to assign dimensions, which may be positive, to individual points, there are now several indications that these dimensions are geometrically meaningful in Euclidean space. For example, results of Hitchcock [7] and Lutz [11] imply that, if $X \subseteq \mathbb{R}^n$ is a union (not necessarily effective) of Π^0_1 (i.e., computably closed) sets, then

$$\dim_{\mathrm{H}}(X) = \sup_{x \in X} \dim(x), \tag{1.1}$$

where $\dim_{\mathrm{H}}(X)$ is the *classical* Hausdorff dimension of X. We thus have a "pointwise" characterization of Hausdorff dimension, which is the most important dimension in fractal geometry, on unions of Π^0_1 sets. Gu, Lutz, and Mayordomo [4] have noted that (1.1), in combination with classical results in geometric measure theory, implies that every point $x \in \mathbb{R}^n$ that lies on a computable curve of finite length has dimension $\dim(x) \leq 1$. For another example, Lutz and Mayordomo [12] have recently carried out a pointwise analysis of the dimensions of self-similar fractals, using

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information-theoretic methods to show that, for every computably self-similar fractal $F \subseteq \mathbb{R}^n$, every point $x \in F$, and every symbolic sequence T that naturally encodes x in the construction of F, the dimension identity

$$\dim(x) = \text{sdim}(F)\dim^{\pi}(T) \tag{1.2}$$

holds. In this equation, $\dim(x)$ is the dimension of the point x in Euclidean space, $\text{sdim}(F)$ is a well known and easily computed quantity called the *similarity dimension* of F [3], and $\dim^{\pi}(T)$ is the dimension of the sequence T with respect to a probability measure π that the fractal F naturally induces on the alphabet of T. (This is a constructive version of Billingsley dimension [1, 2].) The classical theorem of Moran [14], stating that

$$\dim_{\text{H}}(F) = \text{sdim}(F) \tag{1.3}$$

holds for every self-similar fractal F, follows easily from (1.2) by relativization. Considering the dimensions of individual points thus gives a new, information-theoretic proof of (1.3), while at the same time providing additional geometric information about how the dimension is "distributed" in the fractal F and "where it comes from" in the dynamical construction of F.

In this paper we investigate the structures of sets in Euclidean space that are defined in terms of the dimensions of their elements. For each nonempty interval $I \subseteq [0, n]$, we are interested in the *dimension level set*

$$\text{DIM}^I = \{x \in \mathbb{R}^n \mid \dim(x) \in I\}.$$

As will be seen below, known results easily imply that each such set DIM^I is a dense subset of $\mathbb{R}^n$ whose Hausdorff and constructive dimensions are both the supremum of I. Results of Hitchcock, Lutz, and Terwijn [8] imply that (except in the degenerate case $I = [0, 1]$) dimension level sets are somewhat complex, in that they all lie in the second or third level of the arithmetical hierarchy.

Our focus here is on the connectivity properties of the dimension level sets. This is a trivial matter in $\mathbb{R}^1$, so our attention is henceforth directed to Euclidean spaces $\mathbb{R}^n$, where $n \geq 2$. As will be seen, an easy argument shows that the dimension level sets $\text{DIM}^{[0,1)}$ and $\text{DIM}^{(n-1,n]}$ are totally disconnected, i.e., all connected components of these sets are single points.

In contrast, our main theorem shows that the dimension level sets $\text{DIM}^{[0,1]}$ and $\text{DIM}^{[n-1,n]}$ are path-connected, i.e., any two points in one of these sets are connected by a continuous path lying entirely in the set. That is, adding the dimension-1 points to the set $\text{DIM}^{[0,1)}$, or adding the dimension-$(n-1)$ points to the set $\text{DIM}^{(n-1,n]}$, transforms a totally disconnected set into a path-connected set. To prove this theorem, we use geometric properties of Kolmogorov complexity in Euclidean space to develop a theorem relating the dimensions of points that are collinear. This development is itself likely to be useful in future investigations.

The above-described transformations from one extreme of the "connectivity spectrum" to the other are especially intriguing given that, at the time of this writing, we know *nothing* about the connectivity properties of the dimension level sets DIM^1 and DIM^{n-1} that produce these transformations. We do not know whether either of these sets is totally disconnected, and we do not know whether either is path-connected.

2 Kolmogorov Complexity and Constructive Dimension in Euclidean Space

This section summarizes basic elements of Kolmogorov complexity and constructive dimension in Euclidean space that are used in proving our results. Our treatment here is brief and assumes some knowledge of Kolmogorov complexity.

It is convenient to use the (*prefix*) *Kolmogorov complexity*

$$\mathrm{K}(w) = \min\{|\pi| \mid U(\pi) = w\},$$

defined for each string $w \in \{0,1\}^*$, where U is a fixed, optimal prefix Turing machine. (We refer to the standard text by Li and Vitányi [9] for the definitions of prefix Turing machine, optimal such machines, and basic properties of $\mathrm{K}(w)$.) We define the *Kolmogorov complexity* of a natural number $r \in \mathbb{N}$ to be $\mathrm{K}(r) = \mathrm{K}(s_n)$, where $s_0, s_1, s_2, \ldots$ is the standard enumeration of $\{0,1\}^*$. Note that $\mathrm{K}(r) = O(\log r)$ for all $r \in \mathbb{Z}^+$.

Encoding sign bits, numerators, denominators, and tuples, it is straightforward to define, for each $n \in \mathbb{Z}^+$, a coding function $\mathrm{rat}^{(n)} : \{0,1\}^* \xrightarrow{\text{onto}} \mathbb{Q}^n$. We then define the *Kolmogorov complexity* of a rational point $q \in \mathbb{Q}^n$ in the Euclidean space $\mathbb{R}^n$ to be $\mathrm{K}(q) = \min\left\{\mathrm{K}(w) \;\middle|\; \mathrm{rat}^{(n)}(w) = q\right\}$. Standard techniques show that this quantity is, up to the usual additive constant, independent of the precise choice of the coding function $\mathrm{rat}^{(n)}$.

We define the *Kolmogorov complexity* of a point $x \in \mathbb{R}^n$ at *precision* $r \in \mathbb{N}$ to be

$$\mathrm{K}_r(x) = \min\left\{\mathrm{K}(q) \;\middle|\; q \in \mathcal{B}(x, 2^{-r})\right\},$$

where

$$\mathcal{B}(x,\rho) = \{y \in \mathbb{R}^n \mid |x-y| \leq \rho\}$$

is the closed ball of radius ρ about x. This is the minimum length of any program $\pi \in \{0,1\}^*$ for which $U(\pi) \in \mathbb{Q}^n \cap \mathcal{B}(x, 2^{-r})$. We also mention the related quantity

$$\mathrm{K}_r(r,x) = \min\left\{\mathrm{K}(r,q) \;\middle|\; q \in \mathbb{Q}^n \cap \mathcal{B}(x, 2^{-r})\right\},$$

in which the program π must output the precision parameter r in addition to a rational approximation q of x to within 2^{-r}. Using standard techniques, it is easy to verify that there is a constant $a \in \mathbb{N}$ such that, for all $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$,

$$\mathrm{K}_r(x) - a \leq \mathrm{K}_r(r,x) \leq \mathrm{K}_r(x) + \mathrm{K}(r) + b. \tag{2.1}$$

Lutz and Mayordomo [12] have also shown there is a constant $c \in \mathbb{N}$ such that, for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, all $r \in \mathbb{N}$, and all $w_1, \ldots, w_n \in \{0,1\}^r$ such that each w_i is a prefix of a binary expansion of the fractional part $x_i - \lfloor x_i \rfloor$ of x_i, we have

$$|\mathrm{K}_r(r,x) - \mathrm{K}(\lfloor x \rfloor, w_1 w_2 \ldots w_n)| \leq c. \tag{2.2}$$

The following characterization of the (constructive) dimension $\dim(x)$ of each point $x \in \mathbb{R}^n$ is the only property of $\dim(x)$ that we use in this paper. The reader may reasonably either regard this as the definition of $\dim(x)$ or consult the papers [11, 12] for the development of $\dim x$ as a constructive version of classical Hausdorff dimension.

Theorem 2.1. *[12]. For all $x \in \mathbb{R}^n$,*

$$\dim(x) = \liminf_{r\to\infty} \frac{\mathrm{K}_r(x)}{r}.$$

By Theorem 2.1 and elementary properties of Kolmogorov complexity, for all $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$, and all permutations π of $\{1, \ldots, n\}$, we have

$$0 \leq \dim(x) \leq n, \tag{2.3}$$

$$\max\{\dim(x), \dim(y)\} \leq \dim(x, y) \leq \dim(x) + \dim(y), \tag{2.4}$$

and

$$\dim(x) = \dim(\pi(x)), \tag{2.5}$$

where we write $\pi(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. These facts imply that, for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \leq i \leq n$,

$$\dim(x_i) \leq \dim(x) \leq n - 1 + \dim(x_i). \tag{2.6}$$

A point $x \in \mathbb{R}^n$ is *random* if there is a constant $d \in \mathbb{N}$ (which may depend on x) such that, for all $r \in \mathbb{N}$, $\mathrm{K}_r(r, x) \geq nr - d$. It is well known that almost every point in $\mathbb{R}^n$ is random, i.e., that the set of nonrandom points form a set of Lebesgue measure 0 in $\mathbb{R}^n$ [9].

Given a point $x^{(i)} \in \mathbb{R}^{n_i}$ for each $0 \leq i < k$, we say that the points $x^{(0)}, \ldots, x^{(k-1)}$ are *independently random* if the point $(x^{(0)}, \ldots, x^{(k-1)}) \in \mathbb{R}^{n_0 + \cdots + n_{k-1}}$ is random.

Given a point $x^{(i)} \in \mathbb{R}^{n_i}$ for each $i \in \mathbb{N}$, we say that the points $x^{(0)}, x^{(1)}, x^{(2)}, \cdots$ are *independently random* if the points $x^{(0)}, \ldots, x^{(k-1)}$ are independently random for every $k \in \mathbb{Z}^+$.

It is easy to see that $\dim(x) = 0$ for all computable points $x \in \mathbb{R}^n$ and $\dim(x) = n$ for all random points $x \in \mathbb{R}^n$. It follows by (2.6) that, for all $x = (x_1, \ldots, x_n)$ and $1 \leq i \leq n$,

$$x_i \text{ is computable} \implies \dim(x) \leq n - 1 \tag{2.7}$$

and

$$x_i \text{ is random} \implies \dim(x) \geq 1. \tag{2.8}$$

We also use relativized Kolmogorov complexity and dimension in Euclidean space. An *oracle* for a point $x \in \mathbb{R}^n$ is any function $g : \mathbb{N} \to \mathbb{Q}^n$ such that $|g(r) - x| \leq 2^{-r}$ holds for all *queries* $r \in \mathbb{N}$. If we write $\mathrm{Or}(x)$ for the set of all oracles for x, then we define the *Kolmogorov complexity* of a point $y \in \mathbb{R}^n$ at *precision* $r \in \mathbb{N}$ *relative to* the point $x \in \mathbb{R}^n$ to be

$$\mathrm{K}_r^x(y) = \sup_{g \in \mathrm{Or}(x)} \mathrm{K}_r^g(y), \tag{2.9}$$

where $\mathrm{K}_r^g(y)$ denotes the Kolmogorov complexity of y at precision r relative to the specific oracle g. Similarly, the *dimension* of a point $y \in \mathbb{R}^n$ *relative to* the point $x \in \mathbb{R}^n$ is

$$\dim^x(y) = \sup_{g \in \mathrm{Or}(x)} \dim^g(y), \tag{2.10}$$

where $\dim^g(y)$ is the dimension of y relative to g. Definitions (2.9) and (2.10) use the supremum over all g to ensure that $\mathrm{K}_r^x(y)$ and $\dim^x(y)$ depend only upon x, y, and r, i.e., that they cannot be artificially reduced by extra information in any particular oracle $g \in \mathrm{Or}(x)$.

It is routine to verify that all the properties of Kolmogorov complexity and dimension that we have discussed continue to hold when relativized to any point $x \in \mathbb{R}^n$. It is also easy to see that $\dim^x(y) = \dim(y)$ whenever x is computable. A well-known theorem of van Lambalgen [15, 16] implies that $\dim^x(y) = \dim(y) = n$ whenever x and y are independently random.

3 Dimension Level Sets

The main theorem of this paper concerns dimension level sets, which are defined as follows.

Definition. The *dimension level set* given by a set $I \subseteq [0, n]$ is the set

$$\mathrm{DIM}^I = \{x \in \mathbb{R}^n \mid \dim(x) \in I\}.$$

For $\alpha \in [0, n]$, we use the abbreviations $\mathrm{DIM}^\alpha = \mathrm{DIM}^{\{\alpha\}}$, $\mathrm{DIM}^{<\alpha} = \mathrm{DIM}^{[0,\alpha)}$, $\mathrm{DIM}^{\leq\alpha} = \mathrm{DIM}^{[0,\alpha]}$, $\mathrm{DIM}^{>\alpha} = \mathrm{DIM}^{(\alpha,n]}$, and $\mathrm{DIM}^{\geq\alpha} = \mathrm{DIM}^{[\alpha,n]}$.

It was shown in [11] that, for each $\alpha \in [0, n]$, the level sets $\mathrm{DIM}^{<\alpha}$, $\mathrm{DIM}^{\leq\alpha}$, and DIM^α all have both constructive dimension and Hausdorff dimension α. The following fact follows readily.

Theorem 3.1. *For all $\varnothing \neq I \subseteq [0, n]$, $\dim(\mathrm{DIM}^I) = \dim_{\mathrm{H}}(\mathrm{DIM}^I) = \sup I$.*

We now turn to the connectivity properties of the dimension level sets.

Theorem 3.2. *The sets $\mathrm{DIM}^{<1}$ and $\mathrm{DIM}^{>n-1}$ are totally disconnected.*

Proof. Let x and y be distinct elements of $\mathcal{D}$, where $\mathcal{D}$ is either $\mathrm{DIM}^{<1}$ or $\mathrm{DIM}^{>n-1}$. Fix $i \in \{1, \ldots, n\}$ such that $x_i \neq y_i$, and assume without loss of generality that $x_i < y_i$. If $\mathcal{D} = \mathrm{DIM}^{<1}$ let θ be a random real number such that $x_i < \theta < y_i$. If $\mathcal{D} = \mathrm{DIM}^{>n-1}$, let θ be a rational number such that $x_i < \theta < y_i$. In either case, define the $(n-1)$-dimensional hyperplane

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid x_i = \theta\}$$

and the open half-spaces

$$\mathcal{H}^- = \{x \in \mathbb{R}^n \mid x_i < \theta\},$$
$$\mathcal{H}^+ = \{x \in \mathbb{R}^n \mid x_i > \theta\}.$$

By (2.7) and (2.8), $\mathcal{P} \cap \mathcal{D} = \varnothing$. Hence, $\mathcal{H}^-$ and $\mathcal{H}^+$ are open sets in $\mathbb{R}^n$ with $x \in \mathcal{H}^-$, $y \in \mathcal{H}^+$, and $\mathcal{D} \subseteq \mathcal{H}^- \cup \mathcal{H}^+$. This shows that x and y lie in distinct connected components of $\mathcal{D}$. Since x and y are arbitrary distinct points in $\mathcal{D}$ here, this shows that $\mathcal{D}$ is totally disconnected. $\square$

In contrast with Theorem 3.2, we will show that the sets $\mathrm{DIM}^{\leq 1}$ and $\mathrm{DIM}^{\geq n-1}$ are path-connected. Our proof uses the following geometric lemma.

Lemma 3.3. *Let $x, y, z \in \mathbb{R}^n$ be distinct, collinear points. If*

$$a > 1 - \log|x - z| + \max\{0, \log(|x - y| + |y - z|)\},$$

then the following two conditions hold for all $r \in \mathbb{N}$.

1. *$\mathcal{B}(x, 2^{-(r+a)}) \cap \mathcal{B}(z, 2^{-(r+a)}) = \varnothing$.*
2. *Every line that meets $\mathcal{B}(x, 2^{-(r+a)})$ and $\mathcal{B}(z, 2^{-(r+a)})$ also meets $\mathcal{B}(y, 2^{-(r+1)})$.*

Proof. Assume the hypothesis. Then we have

$$a > 1 - \log|x - z| \tag{3.1}$$

and

$$a > 1 + \log(|x - y| + |y - z|) - \log|x - z|. \tag{3.2}$$

By the triangle inequality, (3.2) implies that

$$a > 1. \tag{3.3}$$

Let $r \in \mathbb{N}$.

By (3.1), we have

$$2^{-(r+a)} \leq 2^{-a} < \frac{1}{2}|x - z|,$$

so part 1 of the lemma holds.

To prove part 2 of the lemma, let $\mathcal{L}$ be a line that meets $\mathcal{B}(x, 2^{-(r+a)})$ and $\mathcal{B}(z, 2^{-(r+a)})$. Fix points $q_x \in \mathcal{L} \cap \mathcal{B}(x, 2^{-(r+a)})$ and $q_z \in \mathcal{L} \cap \mathcal{B}(z, 2^{-(r+a)})$. Since x, y, and z are distinct, collinear points, we have the following two cases.

Case 1. y is between x and z. Then there exists $\alpha \in (0,1)$ such that $y = \alpha x + (1-\alpha)z$. Let

$$q_y = \alpha q_x + (1-\alpha)q_z,$$

noting that $q_y \in \mathcal{L}$. By (3.3), we have $q_x \in \mathcal{B}(x, 2^{-(r+1)})$ and $q_z \in \mathcal{B}(z, 2^{-(r+1)})$, so

$$\begin{aligned}
|q_y - y| &= |\alpha q_x + (1-\alpha)q_z - \alpha x - (1-\alpha)z| \\
&\leq \alpha|q_x - x| + (1-\alpha)|q_z - z| \\
&\leq \alpha 2^{-(r+1)} + (1-\alpha)2^{-(r+1)} \\
&= 2^{-(r+1)},
\end{aligned}$$

so $q_y \in \mathcal{L} \cap \mathcal{B}(y, 2^{-(r+1)})$.

Case 2. y is not between x and z. Then, by symmetry, we can assume without loss of generality that z is between x and y. Then there exists $\beta \in (0,1)$ such that $z \in \beta x + (1-\beta)y$. Let

$$q_y = \frac{1}{1-\beta}(q_z - \beta q_x),$$

noting that $q_y \in \mathcal{L}$. Then

$$\begin{aligned}
|q_y - y| &= \left|\frac{1}{1-\beta}(q_z - \beta q_x) - \frac{1}{1-\beta}(z - \beta x)\right| \\
&\leq \frac{1}{1-\beta}(|q_z - z| + \beta|q_x - x|) \\
&\leq \frac{1}{1-\beta}(2^{-(r+a)} + \beta 2^{-(r+a)}) \\
&= 2^{-(r+a)}\frac{1+\beta}{1-\beta} \\
&= 2^{-(r+a)}\frac{1 + \frac{|y-z|}{|y-x|}}{1 - \frac{|y-z|}{|y-x|}} \\
&= 2^{-(r+a)}\frac{|x-y| + |y-z|}{|x-y| - |y-z|} \\
&= 2^{-(r+a)}\frac{|x-y| + |y-z|}{|x-z|} \\
&< 2^{-(r+1)}.
\end{aligned}$$

(The last inequality holds by (3.2).) Hence, $q_y \in \mathcal{L} \cap \mathcal{B}(y, 2^{-(r+1)})$.

In either case, we have shown that $\mathcal{L}$ meets $\mathcal{B}(y, 2^{-(r+1)})$, confirming part 2 of the lemma. $\square$

We next extract the following Kolmogorov complexity result from Lemma 3.3.

Lemma 3.4. *If $x, y, z \in \mathbb{R}^n$ are distinct, collinear points, then there exist constants $a, b \in \mathbb{N}$ such that, for all $r \in \mathbb{N}$,*

$$\mathrm{K}^x_r(y) \leq \mathrm{K}_{r+a}(z) + r + \mathrm{K}(r) + b.$$

Proof. Assume the hypothesis. Let $a \in \mathbb{N}$ satisfy the hypothesis of Lemma 3.3, and let $l = \max\{1, \log|y - z|\}$.

Let M be an oracle prefix Turing machine with the following behavior. Assume that $\pi, \pi' \in \{0,1\}^*$ are programs that cause the optimal prefix Turing machine U to produce outputs $U(\pi) = q \in \mathbb{Q}^n$ and $U(\pi') = r \in \mathbb{N}$. Then, for all $w \in \{0,1\}^{r+l+2}$ and $d \in \{0,1\}$, $M^x(\pi\pi' wd)$ obtains an approximation $q_x \in \mathbb{Q}^n \cap \mathcal{B}(x, 2^{-(r+a)})$ from its oracle and, if $q_x \neq q$, outputs the point

$$q^* = q + (-1)^d 2^{-(r+1)} i_w \frac{q_x - q}{|q_x - q|} \in \mathbb{Q}^n,$$

where w is the $(r + l + 2)$-bit binary representation of $i_w \in \mathbb{N}$. Let $b = 3 + l + c_M$, where c_M is an optimality constant for M.

Let $\pi, \pi' \in \{0,1\}^*$ be programs testifying to the values of $\mathrm{K}_{r+a}(z)$ and $\mathrm{K}(r)$, respectively, and let $q = U(\pi)$. Then $q \in \mathcal{B}(z, 2^{-(r+a)})$, so part 1 of Lemma 3.3 assures us that the point q_x obtained by M will satisfy $q_x \neq q$. Let $\mathcal{L}$ be the line through q and q_x. Then $\mathcal{L}$ meets $\mathcal{B}(x, 2^{-(r+a)})$ and $\mathcal{B}(z, 2^{-(r+a)})$, so part 2 of Lemma 3.3 tells us that $\mathcal{L}$ meets $\mathcal{B}(y, 2^{-(r+1)})$. Since the points

$$q(,d) = q + (-1)^d 2^{-(r+1)} i \frac{q_x - q}{|q_x - q|} \in \mathbb{Q}^n,$$

for $i \in \mathbb{N}$ and $d \in \{0,1\}$, lie along $\mathcal{L}$ at intervals of length $2^{-(r+1)}$, it follows that there exist $i \in \mathbb{N}$ and $d \in \{0,1\}$ such that $q(i,d) \in \mathcal{B}(y, 2^{-r})$. We then have

$$\begin{aligned} 2^{-(r+1)} i &= |q(i,d) - q)| \\ &\leq |y - z| + |q(i,d) - y| + |q - z| \\ &\leq 2^l + 2^{-r} + 2^{-(r+a)} \\ &< 2^{l+1}, \end{aligned}$$

so $i < 2^{r+l+2}$. It follows that there is a string $w \in \{0,1\}^{r+l+2}$ such that $M(\pi\pi' wd) = q(i,d)$. Since $q(i,d) \in \mathcal{B}(y, 2^{-r})$, it follows that

$$\begin{aligned} \mathrm{K}^x_r(y) &\leq \mathrm{K}^x_{M,r}(y) + c_M \\ &\leq |\pi\pi' wd| + c_M \\ &= \mathrm{K}_{r+a}(z) + r + \mathrm{K}(r) + b. \end{aligned}$$

$\square$

We now prove the following useful theorem on the dimensions of collinear points.

Theorem 3.5. *If $x, y, z \in \mathbb{R}^n$ are collinear points with $x \neq z$, then*

$$\dim^x(y) \leq \dim(z) + 1.$$

Proof. Assume the hypothesis. We have three cases.

1. If $y = x$, then $\dim^x(y) = 0 < \dim(z) + 1$.

2. If $y = z$, then $\dim^x(y) \leq \dim(z) < \dim(z) + 1$.

3. If $y \neq x$ and $y \neq z$, then, by Lemma 3.4 and Theorem 2.1, there exist constants $a, b \in \mathbb{N}$ such that

$$\begin{aligned}
\dim^x(y) &= \liminf_{r\to\infty} \frac{\mathrm{K}_r^x(y)}{r} \\
&\leq \liminf_{r\to\infty} \frac{\mathrm{K}_{r+a}(z) + r + \mathrm{K}(r) + b}{r} \\
&= 1 + \liminf_{r\to\infty} \frac{\mathrm{K}_{r+a}(z)}{r} \\
&= 1 + \liminf_{r\to\infty} \frac{\mathrm{K}_r(z)}{r-a} \\
&= 1 + \liminf_{r\to\infty} \frac{\mathrm{K}_r(z)}{r} \\
&= \dim(z) + 1.
\end{aligned}$$

□

Our main theorem uses the following two corollaries of Theorem 3.5.

Corollary 3.6. *If $\mathcal{L}$ is a line through two computable points, then $\mathcal{L} \subseteq \mathrm{DIM}^{\leq 1}$.*

Proof. Assume the hypothesis. Then there exist computable points $x, z \in \mathcal{L}$ such that $x \neq z$. To see that $\mathcal{L} \subseteq \mathrm{DIM}^{\leq 1}$, let $y \in \mathcal{L}$. Then, by Theorem 3.5 and the computability of x and z,

$$\dim(y) = \dim^x(y) \leq \dim(z) + 1 = 1.$$

□

We note that Corollary 3.6 is already known, because it follows from the fact, noted by Gu, Lutz, and Mayordomo [4], that every point on every rectifiable computable curve (i.e., every computable curve of finite length) has dimension at most 1. The direct proof above is nevertheless instructive.

Corollary 3.7. *If $\mathcal{L}$ is a line through two independently random points in $\mathbb{R}^n$, then $\mathcal{L} \subseteq \mathrm{DIM}^{\geq n-1}$.*

Proof. Assume the hypothesis. Then there exist independently random points $x, y \in \mathcal{L}$. To see that $\mathcal{L} \subseteq \mathrm{DIM}^{\geq n-1}$, let $z \in \mathcal{L}$. We have two cases.

1. If $z = x$, then $\dim(z) = \dim(x) = n$.

2. If $z \neq x$, then, by Theorem 3.5 and the independent randomness of x and y,

$$\dim(z) \geq \dim^x(y) - 1 = n - 1.$$

In either case, $\dim(z) \geq n-1$. $\square$

We now have the machinery to prove our main theorem.

Theorem 3.8. *The sets* $\mathrm{DIM}^{\leq 1}$ *and* $\mathrm{DIM}^{\geq n-1}$ *are path-connected.*

Proof. Let $\mathcal{D}$ be either $\mathrm{DIM}^{\leq 1}$ or $\mathrm{DIM}^{\geq n-1}$. To see that $\mathcal{D}$ is path-connected, let $x, y \in \mathcal{D}$. If $\mathcal{D} = \mathrm{DIM}^{\leq 1}$, let $p_0, p_1, \ldots$ and $q_0, q_1, \ldots$ be sequences of points in $\mathbb{Q}^n$ converging to x and y, respectively. If $\mathcal{D} = \mathrm{DIM}^{\geq n-1}$, let $p_0, p_1, \ldots$ and $q_0, q_1, \ldots$ be sequences of points in $\mathbb{R}^n$ converging to x and y, respectively, such that the points $p_0, q_0, p_1, q_1, \ldots$ are independently random. Define a function $f : [0,1] \to \mathbb{R}^n$ as follows.

(i) For each $0 < t \leq \frac{1}{3}$, fix $m(t) \in \mathbb{N}$ such that $\frac{1}{m(t)+4} < t \leq \frac{1}{m(t)+3}$, and set $f(t) = \alpha p_{m(t)+1} + (1-\alpha)p_{m(t)}$, where $t = \frac{\alpha}{m(t)+4} + \frac{(1-\alpha)}{m(t)+3}$.

(ii) For each $\frac{2}{3} \leq t < 1$, fix $m(t) + \mathbb{N}$ such that $1 - \frac{1}{m(t)+3} \leq t < 1 - \frac{1}{m(t)+4}$, and set $f(t) = \alpha q_{m(t)} + (1-\alpha)q_{m(t)+1}$, where

(a) $f(0) = x$.

(b) On each interval $[\frac{1}{m+4}, \frac{1}{m+3}]$, for $m \in \mathbb{N}$, f is a straight-line path from p_{m+1} to p_m.

(c) On the interval $[\frac{1}{3}, \frac{2}{3}]$, f is a straight-line path from p_0 to q_0.

(d) On each interval $[\frac{m+2}{m+3}, \frac{m+3}{m+4}]$, for $m \in \mathbb{N}$, f is a straight-line path from q_m to q_{m+1}.

(e) $f(1) = y$.

Then f is continuous, so $\Gamma = \mathrm{range}(f)$ is a path from x to y. If $\mathcal{D} = \mathrm{DIM}^{\leq 1}$, then $\Gamma \subseteq \mathcal{D}$ by Corollary 3.6. If $\mathcal{D} = \mathrm{DIM}^{\geq n-1}$, then $\Gamma \subseteq \mathcal{D}$ by Corollary 3.7. $\square$

We note that the path-connectedness of $\mathrm{DIM}^{\leq 1}$ is a quantitative extension of the theorem of Hertling and Weihrauch [5] stating that the set of non-random points in $\mathbb{R}^n$ ($n \geq 2$) is path-connected.

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