The Arithmetical Complexity of Dimension and Randomness

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Abstract

Constructive dimension and constructive strong dimension are effectivizations of the Hausdorff and packing dimensions, respectively. Each infinite binary sequence A is assigned a dimension $\dim(A) \in [0,1]$ and a strong dimension $\mathrm{Dim}(A) \in [0,1]$.

Let DIM^α and $\mathrm{DIM}^\alpha_{\mathrm{str}}$ be the classes of all sequences of dimension α and of strong dimension α, respectively. We show that DIM^0 is properly Π^0_2, and that for all Δ^0_2-computable $\alpha \in (0,1]$, DIM^α is properly Π^0_3.

To classify the strong dimension classes, we use a more powerful effective Borel hierarchy where a co-enumerable predicate is used rather than a enumerable predicate in the definition of the Σ^0_1 level. For all Δ^0_2-computable $\alpha \in [0,1)$, we show that $\mathrm{DIM}^\alpha_{\mathrm{str}}$ is properly in the Π^0_3 level of this hierarchy. We show that $\mathrm{DIM}^1_{\mathrm{str}}$ is properly in the Π^0_2 level of this hierarchy.

We also prove that the class of Schnorr random sequences and the class of computably random sequences are properly Π^0_3.

Keywords: arithmetical hierarchy, constructive dimension, Schnorr randomness, computable randomness

1 Introduction

Hausdorff dimension – the most extensively studied fractal dimension – has recently been effectivized at several levels of complexity, yielding applications to a variety of topics in theoretical computer science, including data compression, polynomial-time degrees, approximate optimization, feasible prediction, circuit-size complexity, Kolmogorov complexity, and randomness [13, 14, 3, 1, 8, 5, 7, 16]. The most fundamental of these effectivizations is *constructive dimension*, which is closely related to Kolmogorov complexity and algorithmic randomness. For every subset $\mathcal{X}$ of $\mathbf{C}$, the *Cantor space* of all infinite binary sequences, a constructive dimension $\mathrm{cdim}(\mathcal{X}) \in [0,1]$ is assigned. Informally, this dimension is determined by the maximum rate of growth that a lower semicomputable martingale can achieve on all sequences in $\mathcal{X}$.

Just as Martin-Löf [15] used constructive measure to define the randomness of individual sequences, Lutz [14] used constructive dimension to define the dimensions of individual sequences. Each sequence $A \in \mathbf{C}$ is assigned a *dimension* $\dim(A) \in [0,1]$ by $\dim(A) = \mathrm{cdim}(\{A\})$. Every

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Martin-Löf random sequence has dimension 1, but there are nonrandom sequences with dimension 1. For every real number $\alpha \in [0,1]$, there is a sequence with dimension α.

It is useful to understand the arithmetical complexity of defining a class of sequences. For example, knowing that RAND, the class of Martin-Löf random sequences, is a Σ^0_2-class allows the application of Kreisel's Basis Lemma [11, 17] to give a short proof [21] that

$$\mathrm{RAND} \cap \Delta^0_2 \neq \emptyset. \tag{1.1}$$

For any $\alpha \in [0,1]$, let

$$\mathrm{DIM}^\alpha = \{A \in \mathbf{C} \mid \dim(A) = \alpha\}.$$

Lutz [14] showed that

$$\mathrm{DIM}^\alpha \cap \Delta^0_2 \neq \emptyset \tag{1.2}$$

for any Δ^0_2-computable $\alpha \in [0,1]$. As these dimension classes do not appear to be Σ^0_2, Lutz was unable to apply the Basis Lemma to them, so he used different techniques to prove (1.2).

We investigate the complexities of these dimension classes in terms of the arithmetical hierarchy of subsets of $\mathbf{C}$. For all Δ^0_2-computable $\alpha \in (0,1]$, we show that DIM^α is properly Π^0_3. Therefore, the proof for (1.1) using Kreisel's Basis Lemma cannot directly be used to establish (1.2). We also show that DIM^0 is properly Π^0_2.

More recently, packing dimension, another important fractal dimension, has also been effectivized by Athreya, Hitchcock, Lutz, and Mayordomo [2]. At the constructive level, this is used in an analogous way to define the *strong dimension* $\mathrm{Dim}(A) \in [0,1]$ for every sequence A. For any $\alpha \in [0,1]$, let

$$\mathrm{DIM}^\alpha_{\mathrm{str}} = \{A \in \mathbf{C} \mid \mathrm{Dim}(A) = \alpha\}.$$

To classify these strong dimension classes, we use introduce a more powerful effective Borel hierarchy where a co-enumerable predicate is used rather than a enumerable predicate in the definition of the Σ^0_1 level. We show that $\mathrm{DIM}^1_{\mathrm{str}}$ is properly in the Π^0_2 level of this stronger hierarchy. For all Δ^0_2-computable $\alpha \in [0,1)$, we show that $\mathrm{DIM}^\alpha_{\mathrm{str}}$ is properly in the Π^0_3 level of this hierarchy.

Our techniques for classifying the dimension and strong dimension classes include Baire category, Wadge reductions, and Kolmogorov complexity.

We conclude the paper with a section on effective randomness classes. We restate a result of Schnorr [19] concerning computable null sets of exponential order in terms of *computable dimension* and point out a relationship with *Church randomness*. We prove that the class of *Schnorr random* sequences and that the class of *computably random* sequences are properly Π^0_3.

Section 2 gives an overview of the randomness and dimension notions used in this paper. In Section 3 we introduce the stronger effective Borel hierarchy that we use for the strong dimension classes. Section 4 presents the classification of DIM^α and $\mathrm{DIM}^\alpha_{\mathrm{str}}$. We conclude with Section 5 on randomness classes.

2 Background on Randomness and Dimension

This section provides an overview of the notions of randomness and dimension used in this paper. We write $\{0,1\}^*$ for the set of all finite binary *strings* and $\mathbf{C}$ for the *Cantor space* of all infinite binary *sequences*. In the standard way, a sequence $A \in \mathbf{C}$ can be identified with the subset of $\{0,1\}^*$ or $\mathbb{N}$ for which it is the characteristic sequence, or with a real number in the unit interval. The length of a string $w \in \{0,1\}^*$ is $|w|$. The string consisting of the first n bits of $x \in \{0,1\}^* \cup \mathbf{C}$ is denoted by $x{\upharpoonright}n$. We write $w \sqsubseteq x$ if w is a prefix of A.

2.1 Martin-Löf Randomness

Martin-Löf [15] introduced the notion of a *constructive null set*. A set is constructively null if it can be covered by a uniform sequence of c.e. open sets that are shrinking in size. That is, $\mathcal{A} \subseteq \mathbf{C}$ is constructive null if $\mathcal{A} \subseteq \bigcap_i \mathcal{U}_i$, where $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ is uniformly c.e. such that $\mu(\mathcal{U}_i) \leq 2^{-i}$. The sequence $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ is called a *Martin-Löf test*. An individual sequence $A \in \mathbf{C}$ is *Martin-Löf random* if $\{A\}$ is not constructively null. The Martin-Löf random sequences play an important role in algorithmic information theory, see e.g. Li and Vitányi [12].

Schnorr [19], following Ville [22], characterized constructive null sets in terms of martingales. A function $d : \{0,1\}^* \to [0,\infty)$ is a *martingale* if for every $w \in \{0,1\}^*$, d satisfies the averaging condition

$$2d(w) = d(w0) + d(w1),$$

and d is a *supermartingale* if it satisfies

$$2d(w) \geq d(w0) + d(w1).$$

The *success set of* d is

$$S^\infty[d] = \left\{ A \in \mathbf{C} \,\middle|\, \limsup_{n\to\infty} d(A{\restriction}n) = \infty \right\},$$

i.e., it is the set of all sequences on which d has unbounded value. We say that d succeeds on a class $\mathcal{A} \subseteq \mathbf{C}$ if $\mathcal{A} \subseteq S^\infty[d]$.

Ville [22] proved that a set $\mathcal{A} \subseteq \mathbf{C}$ has Lebesgue measure 0 if and only if there is a martingale d that succeeds on $\mathcal{A}$. Schnorr [19] showed that $\mathcal{A}$ is constructively null if and only if d can be chosen to be lower semicomputable, that is, if d can be computably approximated from below. We call such a d *constructive*.

Martin-Löf [15] proved that there is a universal constructive null set. That is, he proved that there is a Martin-Löf test $\{\mathcal{U}_i\}_i$ such that for every other test $\{\mathcal{V}_i\}$ it holds that $\bigcap_i \mathcal{V}_i \subseteq \bigcap_i \mathcal{U}_i$. By Schnorr's analysis this implies that there is also a universal constructive supermartingale $\mathbf{d}$. That is, for any constructive supermartingale d' there is a $c > 0$ such that $\mathbf{d}(w) \geq cd'(w)$ for all $w \in \{0,1\}^*$. We will use this universal supermartingale in section 4. We denote the complement of $S^\infty[\mathbf{d}]$ by RAND, so that RAND consists of all the Martin-Löf random sequences.

2.2 Schnorr Randomness

Schnorr [19] criticized the notion of constructive null for an actual lack of constructiveness, and introduced the more constructive notion of a *Schnorr null set*, which is defined by requiring that the measure of the levels $\mathcal{U}_i$ in a Martin-Löf test be computably approximable to within any given precision. It is easy to see that this is equivalent to the following: $\mathcal{A}$ is Schnorr null if $\mathcal{A} \subseteq \bigcap_i \mathcal{U}_i$, where $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ is uniformly c.e. such that $\mu(\mathcal{U}_i) = 2^{-i}$. The sequence $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ is called a *Schnorr test*.

Following Schnorr [19], we call an unbounded nondecreasing function $h : \{0,1\}^* \to \{0,1\}^*$ an *order*. (N.B. An "Ordnungsfunktion" in Schnorr's terminology is always computable, whereas we prefer to leave the complexity of orders unspecified in general.) For any order h and martingale d, we define the *order* h *success set of* d as

$$S^h[d] = \left\{ A \in \mathbf{C} \,\middle|\, \limsup_{n\to\infty} \frac{d(A{\restriction}n)}{h(n)} \geq 1 \right\}.$$

Schnorr pointed out that the rate of success of a constructive martingale d can be so slow that it cannot be computably detected. Thus rather than working with constructive null sets of the form $S^{\infty}[d]$ with d constructive, he worked with null sets of the form $S^{h}[d]$, where both d and h are computable. He proved that a set $\mathcal{A}$ is Schnorr null if and only if it is included in a null set of the form $S^{h}[d]$, with d and h computable.

A sequence $A \in \mathbf{C}$ is *Schnorr random* if $\{A\}$ is not Schnorr null. This is related the notion of computable randomness. A sequence A is *computably random* if for any computable martingale d, $A \notin S^{\infty}[d]$.

We write $\mathrm{RAND}_{\mathrm{Schnorr}}$ for the class of all Schnorr random sequences and $\mathrm{RAND}_{\mathrm{comp}}$ for the class of all computably random sequences. By definition we have that

$$\mathrm{RAND} \subseteq \mathrm{RAND}_{\mathrm{comp}} \subseteq \mathrm{RAND}_{\mathrm{Schnorr}}.$$

The first inclusion was proved strict by Schnorr [19], and the second inclusion was proved strict by Wang [23].

2.3 Constructive Dimension

Hausdorff [6], introduced the concept of null covers that "succeed exponentially fast" to define what is now commonly called Hausdorff dimension, the most widely used dimension in fractal geometry. Basically, this notion allows one to discern structure in classes of measure zero, and to calibrate them. As for constructive measure, already Schnorr (see Theorem 5.1) drew special attention to null sets of "exponential order", although he did not make an explicit connection to Hausdorff dimension.

Lutz [13, 14] gave a characterization of Hausdorff dimension in terms of *gales*, which are a generalization of martingales. Let $s \in [0, \infty)$. An *s-gale* is a function $d : \{0,1\}^* \to [0,\infty)$ that satisfies the averaging condition

$$2^{s} d(w) = d(w0) + d(w1) \tag{2.1}$$

for every $w \in \{0,1\}^*$. Similarly, d is an *s-supergale* if (2.1) holds with $\geq$ instead of equality. The success set $S^{\infty}[d]$ is defined exactly as was done for martingales above. Lutz showed that for any class $\mathcal{A} \subseteq \mathbf{C}$, the Hausdorff dimension of $\mathcal{A}$ is

$$\dim_{\mathrm{H}}(\mathcal{A}) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there exists an } s\text{-gale} \\ d \text{ for which } \mathcal{A} \subseteq S^{\infty}[d] \end{array} \right\}. \tag{2.2}$$

Lutz [14] effectivized this characterization to define the constructive dimensions of sets and sequences. An s-(super)gale is called *constructive* if it is lower semicomputable. The *constructive dimension* of a class $\mathcal{A} \subseteq \mathbf{C}$ is

$$\mathrm{cdim}(\mathcal{A}) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there exists a constructive } s\text{-gale} \\ d \text{ for which } \mathcal{A} \subseteq S^{\infty}[d] \end{array} \right\} \tag{2.3}$$

and the constructive dimension of an individual sequence $A \in \mathbf{C}$ is

$$\dim(A) = \mathrm{cdim}(\{A\}).$$

(Supergales can be equivalently used in place of gales in both (2.2) and (2.3) [13, 9, 4].)

Constructive dimension has some remarkable properties. For example, Lutz [14] showed that for any class $\mathcal{A}$,

$$\mathrm{cdim}(\mathcal{A}) = \sup_{A \in \mathcal{A}} \dim(A). \tag{2.4}$$

Also, Mayordomo [16] established an strong connection with *Kolmogorov complexity*: for any $A \in \mathbf{C}$,

$$\dim(A) = \liminf_{n\to\infty} \frac{K(A \upharpoonright n)}{n}, \tag{2.5}$$

where $K(A \upharpoonright n)$ is the size of the smallest program that causes a fixed universal self-delimiting Turing machine to output the first n bits of A. (For more details on Kolmogorov complexity, we refer to [12].)

2.4 Constructive Strong Dimension

More recently, Athreya, Hitchcock, Lutz, and Mayordomo [2] also characterized *packing dimension*, another important fractal dimension, in terms of gales. For this, the notion of *strong success* of an s-gale d was introduced. The *strong success set of* d is

$$S^{\infty}_{\mathrm{str}}[d] = \left\{ A \in \mathbf{C} \,\middle|\, \liminf_{n\to\infty} d(A \upharpoonright n) = \infty \right\}.$$

Analogously to what was done for Hausdorff dimension, packing dimension can be characterized using strong success sets of gales. Effectivizing this in the same way leads to the definition of the *constructive strong dimension* of a class $\mathcal{A} \subseteq \mathbf{C}$ as

$$\mathrm{cDim}(\mathcal{A}) = \inf \left\{ s \,\middle|\, \begin{array}{l} \text{there exists a constructive } s\text{-gale} \\ d \text{ for which } \mathcal{A} \subseteq S^{\infty}_{\mathrm{str}}[d] \end{array} \right\}.$$

The constructive strong dimension of a sequence $A \in \mathbf{C}$ is

$$\mathrm{Dim}(A) = \mathrm{cDim}(\{A\}).$$

A pointwise stability property analogous to (2.4) also holds for strong dimension, as well as a Kolmogorov complexity characterization [2]:

$$\mathrm{Dim}(A) = \limsup_{n\to\infty} \frac{K(A \upharpoonright n)}{n} \tag{2.6}$$

for any $A \in \mathbf{C}$.

3 Borel Hierarchies

We use $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ to denote the levels of the Borel hierarchy for subsets of Cantor space. The levels of the arithmetical hierarchy (the corresponding effective hierarchy) are denoted by Σ^0_n and Π^0_n.

We will also make use of the following more general hierarchy definition.

Definition. *Let $\mathcal{P}$ be a class of predicates, let $n \geq 1$, and let $\mathcal{X} \subseteq \mathbf{C}$.*

- $\mathcal{X} \in \Sigma_n^0[\mathcal{P}]$ *if for some predicate* $P \in \mathcal{P}$,

$$A \in \mathcal{X} \iff (\exists k_n)(\forall k_{n-1}) \cdots (Qk_1) P(k_n, \ldots, k_2, A{\upharpoonright}k_1),$$

where $Q = \exists$ *if* n *is odd and* $Q = \forall$ *if* n *is even.*

- $\mathcal{X} \in \Pi_n^0[\mathcal{P}]$ *if for some predicate* $P \in \mathcal{P}$,

$$A \in \mathcal{X} \iff (\forall k_n)(\exists k_{n-1}) \cdots (Qk_1) P(k_n, \ldots, k_2, A{\upharpoonright}k_1),$$

where $Q = \forall$ *if* n *is odd and* $Q = \exists$ *if* n *is even.*

If we take $\mathcal{P}$ to be Δ_1^0 (decidable predicates), then the above definition is equivalent to the standard arithmetical hierarchy; that is,

$$\Sigma_n^0 = \Sigma_n^0[\Delta_1^0]$$

and

$$\Pi_n^0 = \Pi_n^0[\Delta_1^0]$$

hold for all n. Also, if ALL is the class of all predicates, then we obtain the classical Borel hierarchy:

$$\mathbf{\Sigma_n^0} = \Sigma_n^0[\mathrm{ALL}]$$

and

$$\mathbf{\Pi_n^0} = \Pi_n^0[\mathrm{ALL}].$$

In this paper, we will also be interested in the cases where $\mathcal{P}$ is Σ_1^0 (enumerable predicates) or Π_1^0 (co-enumerable predicates). In some cases, the classes in the generalized hierarchy using these sets of predicates are no different that the standard arithmetical hierarchy classes. If n is odd, then $\Sigma_n^0 = \Sigma_n^0[\Sigma_1^0]$ as the existential quantifier in the Σ_1^0 predicate can be absorbed into the last quantifier in the definition of $\Sigma_n^0[\Delta_1^0] = \Sigma_n^0$. Analogously, $\Pi_n^0 = \Pi_n^0[\Pi_1^0]$ for odd n, and for even n we have $\Sigma_n^0 = \Sigma_n^0[\Pi_1^0]$ and $\Pi_n^0 = \Pi_n^0[\Sigma_1^0]$. On the other hand, using the complementary set of predicates defines an effective hierarchy that is distinct from and interleaved with the arithmetical hierarchy.

Proposition 3.1. *1. If* n *is odd, then*

$$\Sigma_n^0 \subsetneq \Sigma_n^0[\Pi_1^0] \subsetneq \Sigma_{n+1}^0$$

and

$$\Pi_n^0 \subsetneq \Pi_n^0[\Sigma_1^0] \subsetneq \Pi_{n+1}^0$$

2. If n *is even, then*

$$\Sigma_n^0 \subsetneq \Sigma_n^0[\Sigma_1^0] \subsetneq \Sigma_{n+1}^0$$

and

$$\Pi_n^0 \subsetneq \Pi_n^0[\Pi_1^0] \subsetneq \Pi_{n+1}^0$$

Proof. We only show $\Sigma_n^0 \subsetneq \Sigma_n^0[\Pi_1^0] \subsetneq \Sigma_{n+1}^0$ for odd n; the arguments for the other statements are analogous.

The inclusion $\Sigma_n^0 \subseteq \Sigma_n^0[\Pi_1^0]$ is true because $\Sigma_n^0 = \Sigma_n^0[\Delta_1^0]$ and $\Delta_1^0 \subseteq \Pi_1^0$. To show that it is proper, let P be a predicate that is complete for the class of Π_n^0 predicates. Then there is a decidable predicate R such that

$$P(n) \iff (\forall k_n)(\exists k_{n-1}) \cdots (\forall k_1) R(n, k_n, \cdots, k_1).$$

Define $\mathcal{X} \subseteq \mathbf{C}$ as

$$\mathcal{X} = \bigcup_{n \in P} 0^n 1\mathbf{C}.$$

Then $\mathcal{X} \in \Sigma_n^0[\Pi_1^0]$ as we have

$$\begin{aligned} S \in \mathcal{X} &\iff (\exists n)P(n) \text{ and } 0^n 1 \sqsubseteq S \\ &\iff (\exists n)(\forall k_n)(\exists k_{n-1}) \cdots (\forall k_1) R(n, k_n, \cdots, k_1) \text{ and } 0^n 1 \sqsubseteq S \\ &\iff (\exists n)(\forall k_n)(\exists k_{n-1}) \cdots (\exists k_2) T(n, k_n, \cdots, k_3, S\restriction k_2), \end{aligned}$$

where T is the Π_1^0 predicate defined by

$$T(n, k_n, \cdots, k_3, w) \iff (\forall k_1) R(n, k_n, \cdots, k_3, |w|, k_1) \text{ and } 0^n 1 \sqsubseteq w.$$

Now suppose that $\mathcal{X} \in \Sigma_n^0$. Then for some decidable predicate U,

$$S \in X \iff (\exists k_n)(\forall k_{n-1}) \cdots (\exists k_1) U(k_n, \cdots, k_2, S\restriction k_1).$$

We then have

$$\begin{aligned} n \in P &\iff 0^n 1\mathbf{C} \subseteq \mathcal{X} \\ &\iff 0^n 10^\infty \in \mathcal{X} \\ &\iff (\exists k_n)(\forall k_{n-1}) \cdots (\exists k_1) U(k_n, \cdots, k_2, 0^n 10^\infty \restriction k_1), \end{aligned}$$

so P is a Σ_3^0 predicate, a contradiction of its Π_3^0-completeness. Therefore $\mathcal{X} \notin \Sigma_3^0$ and we've established $\Sigma_n^0 \subsetneq \Sigma_n^0[\Pi_1^0]$.

The inclusion $\Sigma_n^0[\Pi_1^0] \subseteq \Sigma_{n+1}^0$ is immediate from the definitions using $\Sigma_{n+1}^0 = \Sigma_{n+1}^0[\Delta_1^0]$. That it is proper follows from the facts $\Sigma_{n+1}^0 - \mathbf{\Sigma_n^0} \neq \emptyset$ and $\Sigma_n^0[\Pi_1^0] \subseteq \mathbf{\Sigma_n^0}$. □

Intuitively, the classes $\Sigma_1^0[\Pi_1^0]$, $\Pi_1^0[\Sigma_1^0]$, $\Sigma_2^0[\Sigma_1^0]$, $\Pi_2^0[\Pi_1^0], \ldots$ are slightly more powerful than their respective counterparts in the arithmetical hierarchy because they use one additional quantifier that is limited to the predicate. We now give a simple example of a class that is best classified in this hierarchy: the class of all 1-generic sequences is $\Pi_2^0[\Pi_1^0]$ but not Π_2^0.

Example 3.2. Recall that a sequence $X \in \mathbf{C}$ is 1-generic if

$$(\forall e)(\exists \sigma \sqsubset X)\big[\, \{e\}^\sigma(e) \downarrow \ \vee (\forall \tau \sqsupset \sigma)[\{e\}^\tau(e) \uparrow] \big]$$

From this definition it is immediate that the class $\mathcal{G} = \{X \mid X \text{ is 1-generic}\}$ is in $\Pi_2^0[\Pi_1^0]$. Suppose that $\mathcal{G}$ is Π_2^0. Then there is a computable predicate R such that

$$X \in \mathcal{G} \iff (\forall n)(\exists m)\big[R(n, X\restriction m)\big].$$

As $\mathcal{G}$ is dense, we can now easily construct a computable element of it by a computable finite extension argument. (Given σ at stage n, search for extension $\sigma' \sqsupseteq \sigma$ such that $R(n, \sigma')$. Such extension will be found by density. Take this extension and proceed to stage $n + 1$.)

4 Classification of DIM^α and $\mathrm{DIM}^\alpha_{\mathrm{str}}$

In this section we investigate the arithmetical complexity of the following dimension and strong dimension classes.

$$\begin{aligned}
\mathrm{DIM}^\alpha &= \{A \in \mathbf{C} \mid \dim(A) = \alpha\} \\
\mathrm{DIM}^{\leq\alpha} &= \{A \in \mathbf{C} \mid \dim(A) \leq \alpha\} \\
\mathrm{DIM}^{\geq\alpha} &= \{A \in \mathbf{C} \mid \dim(A) \geq \alpha\} \\
\mathrm{DIM}^\alpha_{\mathrm{str}} &= \{A \in \mathbf{C} \mid \mathrm{Dim}(A) = \alpha\} \\
\mathrm{DIM}^{\leq\alpha}_{\mathrm{str}} &= \{A \in \mathbf{C} \mid \mathrm{Dim}(A) \leq \alpha\} \\
\mathrm{DIM}^{\geq\alpha}_{\mathrm{str}} &= \{A \in \mathbf{C} \mid \mathrm{Dim}(A) \geq \alpha\}
\end{aligned}$$

Let $\alpha \in [0,1]$ be Δ^0_2-computable. For any such α, it is well known that there is a computable function $\hat{\alpha} : \mathbb{N} \to \mathbb{Q}$ such that $\lim\limits_{n\to\infty} \hat{\alpha}(n) = \alpha$. Using (2.5), we have

$$\begin{aligned}
\dim(X) \leq \alpha &\iff \liminf_{n\to\infty} \frac{K(X{\upharpoonright}n)}{n} \leq \alpha \\
&\iff (\forall k)(\forall N)(\exists n \geq N) K(X{\upharpoonright}n) < (\hat{\alpha}(n) + 1/k)n,
\end{aligned}$$

so $\mathrm{DIM}^{\leq\alpha}$ is a Π^0_2-class. Also,

$$\begin{aligned}
\dim(X) \geq \alpha &\iff \liminf_{n\to\infty} \frac{K(X{\upharpoonright}n)}{n} \geq \alpha \\
&\iff (\forall k)(\exists N)(\forall n \geq N) K(X{\upharpoonright}n) > (\hat{\alpha}(N) - 1/k)n,
\end{aligned}$$

so $\mathrm{DIM}^{\geq\alpha}$ is a Π^0_3-class. Therefore we have the following.

Proposition 4.1. *1. The class* DIM^0 *is* Π^0_2.

2. *For all* Δ^0_2*-computable* $\alpha \in (0,1]$, DIM^α *is a* Π^0_3*-class.*

3. *For arbitrary* $\alpha \in (0,1]$, DIM^α *is a* $\boldsymbol{\Pi^0_3}$*-class.*

The situation is slightly more complicated for strong dimension. By (2.6), we have

$$\begin{aligned}
\mathrm{Dim}(X) \leq \alpha &\iff \limsup_{n\to\infty} \frac{K(X{\upharpoonright}n)}{n} \leq \alpha \\
&\iff (\forall k)(\exists N)(\forall n \geq N) K(X{\upharpoonright}n) < (\hat{\alpha}(N) + 1/k)n \\
&\iff (\forall k)(\exists N)(\forall n \geq N)(\exists < \pi, t >)|\pi| < (\hat{\alpha}(N) + 1/k)n \\
&\qquad \text{and } U(\pi) = X{\upharpoonright}n \text{ in } \leq t \text{ computation steps,}
\end{aligned}$$

where U is the fixed universal self-delimiting Turing machine used to define K. From this it is clear that $\mathrm{DIM}^{\leq\alpha}_{\mathrm{str}} \in \Pi^0_4$. However, the "$(\exists < \pi, t >)$" quantifier is local to the defining predicate, so we have $\mathrm{DIM}^{\leq\alpha}_{\mathrm{str}} \in \boldsymbol{\Pi^0_3}$, and in fact, it is a $\Pi^0_3[\Sigma^0_1]$-class. Also,

$$\begin{aligned}
\mathrm{Dim}(X) \geq \alpha &\iff \limsup_{n\to\infty} \frac{K(X{\upharpoonright}n)}{n} \geq \alpha \\
&\iff (\forall k)(\forall N)(\exists n \geq N) K(X{\upharpoonright}n) > (\hat{\alpha}(n) - 1/k)n,
\end{aligned}$$

so $\mathrm{DIM}^{\geq\alpha}_{\mathrm{str}}$ is a $\Pi^0_2[\Pi^0_1]$-class. This establishes the following analogue of Proposition 4.1.

Proposition 4.2. *1. The class* $\mathrm{DIM}^1_{\mathrm{str}}$ *is* $\Pi^0_2[\Pi^0_1]$.

2. *For all* Δ^0_2*-computable* $\alpha \in [0,1)$, $\mathrm{DIM}^\alpha_{\mathrm{str}}$ *is a* $\Pi^0_3[\Sigma^0_1]$*-class.*

3. *For arbitrary* $\alpha \in [0,1)$, $\mathrm{DIM}^\alpha_{\mathrm{str}}$ *is* $\mathbf{\Pi^0_3}$*-class.*

In this remainder of this section we use category methods and Wadge reductions to prove that the classifications in Propositions 4.1 and 4.2 cannot be improved in their respective hierarchies.

4.1 Category Methods

Recall that a class $\mathcal{X}$ is *meager* if it is included in a countable union of nowhere dense subsets of $\mathbf{C}$, and *comeager* if its complement $\overline{\mathcal{X}}$ is meager. The following lemma (implicit in Rogers [18, p341]) will be useful.

Lemma 4.3. *If* $\mathcal{X} \in \mathbf{\Sigma^0_2}$ *and* $\overline{\mathcal{X}}$ *is dense then* $\mathcal{X}$ *is meager.*

Proof. Suppose that $\mathcal{X} = \bigcup_n \mathcal{X}_n$, $\mathcal{X}_n$ closed. Since $\overline{\mathcal{X}}$ is dense, $\mathcal{X}_n$ contains no basic open set, hence $\mathcal{X}_n$ is nondense (i.e. its closure contains no basic open set), and $\mathcal{X}$ is a countable union of nondense sets. □

The class RAND of Martin-Löf random sets can easily be classified with category methods.

Theorem 4.4. (folk) RAND *is a* Σ^0_2*-class, but not a* $\mathbf{\Pi^0_2}$*-class.*

Proof. This is analogous to the proof in Rogers [18, p 341] that $\{X : X \text{ finite}\}$ is a Σ^0_2-class but not a Π^0_2-class. Both RAND and its complement are dense, so by Lemma 4.3, RAND is meager. If RAND were a $\mathbf{\Pi^0_2}$-class, then again using Lemma 4.3, its complement would also be meager. This contradicts the fact that $\mathbf{C}$ is not meager. □

As DIM^0 and $\mathrm{DIM}^1_{\mathrm{str}}$ are dense $\mathbf{\Pi^0_2}$-classes that have dense complements, a similar argument as the one used for Theorem 4.4 shows that they are not $\mathbf{\Sigma^0_2}$-classes.

Theorem 4.5. *The classes* DIM^0 *and* $\mathrm{DIM}^1_{\mathrm{str}}$ *are not* $\mathbf{\Sigma^0_2}$*-classes.*

We now develop category methods for the other DIM^α classes. For every rational s, define the computable order $h_s(n) = 2^{(1-s)n}$. Let $\mathbf{d}$ be the optimal constructive supermartingale.

Lemma 4.6. *For every rational* $s \in (0,1)$, $S^{h_s}[\mathbf{d}]$ *is a comeager* Π^0_2*-class.*

Proof. Notice that $\overline{S^{h_s}[\mathbf{d}]} \in \Sigma^0_2$ and $S^{h_s}[\mathbf{d}]$ is dense. Now apply Lemma 4.3. □

Lemma 4.7. *For all* $\alpha \in (0,1]$, DIM^α *is meager.*

Proof. Let $s < \alpha$ be rational. Lutz [14] showed that $\mathbf{d}^{(s)}(w) = 2^{(s-1)|w|}\mathbf{d}(w)$ is an optimal constructive s-supergale. It follows that for any $A \in \mathbf{C}$, $A \in S^{h_s}[\mathbf{d}] \Rightarrow \dim(S) < \alpha$. Therefore $\mathrm{DIM}^\alpha \subseteq \overline{S^{h_s}}$, so DIM^α is meager by Lemma 4.6. □

Proposition 4.8. *For all* $\alpha \in (0,1]$, DIM^α *is not a* Π^0_2*-class.*

Proof. If $\mathrm{DIM}^\alpha \in \Pi^0_2$, then Lemma 4.3 implies that DIM^α is comeager, contradicting Lemma 4.7. □

To strengthen Proposition 4.8 to show that DIM^α is not Σ^0_3, we now turn to Wadge reductions.

4.2 Wadge Reductions

Let $\mathcal{A}, \mathcal{B} \subseteq \mathbf{C}$. A *Wadge reduction* of $\mathcal{A}$ to $\mathcal{B}$ is a function $f : \mathbf{C} \to \mathbf{C}$ that is continuous and satisfies $\mathcal{A} = f^{-1}(\mathcal{B})$, i.e., $X \in \mathcal{A} \iff f(X) \in \mathcal{B}$. We say that $\mathcal{B}$ is *Wadge complete* for a class $\mathbf{X}$ of subsets of $\mathbf{C}$ if $\mathcal{B} \in \mathbf{X}$ and every $\mathcal{A} \in \mathbf{X}$ Wadge reduces to $\mathcal{B}$. As the classes of the Borel hierarchy are closed under Wadge reductions, Wadge completeness can be used to properly identify the location of a subset of $\mathbf{C}$ in the hierarchy.

We now prove that DIM^1 is Wadge complete for $\mathbf{\Pi^0_3}$. We will then give Wadge reductions from it to DIM^α for the other values of α.

Theorem 4.9. *DIM^1 is Wadge complete for $\mathbf{\Pi^0_3}$. Therefore DIM^1 is not a $\mathbf{\Sigma^0_3}$-class, and in particular is not a Σ^0_3 -class.*

Proof. One could prove this by reducing a known $\mathbf{\Pi^0_3}$-complete class to DIM^1, e.g. the class of sets that have a limiting frequency of 1's that is 0 (this class was proved to be $\mathbf{\Pi^0_3}$-complete by Ki and Linton [10]), but is just as easy to build a direct reduction from an arbitrary $\mathbf{\Pi^0_3}$-class.

Let $\mathbf{d}$ be the universal constructive supermartingale. Note that we have

$$S^{2^n}[\mathbf{d}] \subsetneq \ldots \subsetneq S^{2^{\frac{1}{k}n}}[\mathbf{d}] \subsetneq S^{2^{\frac{1}{k+1}n}}[\mathbf{d}] \subsetneq \ldots \subsetneq \mathrm{DIM}^1.$$

Let $\bigcup_k \bigcap_s \mathcal{O}_{k,s}$ be a $\mathbf{\Sigma^0_3}$-class. Without loss of generality $\mathcal{O}_{k,s} \supseteq \mathcal{O}_{k,s+1}$ for all k,s. We define a continuous function $f : \mathbf{C} \to \mathbf{C}$ such that

$$\forall k \left(X \in \bigcap_s \mathcal{O}_{k,s} \Longleftrightarrow f(X) \in S^{2^{\frac{1}{k}n}}[\mathbf{d}] \right) \tag{4.1}$$

so that we have

$$\begin{aligned} X \notin \bigcup_k \bigcap_s \mathcal{O}_{k,s} &\iff \forall k \left(f(X) \notin S^{2^{\frac{1}{k}n}}[\mathbf{d}] \right) \\ &\iff f(X) \in \mathrm{DIM}^1. \end{aligned}$$

The image $Y = f(X)$ is defined in stages, $Y = \bigcup_s Y_s$, such that every initial segment of X defines an initial segment of Y.

At stage 0 we define Y_0 to be the empty sequence.

At stage $s > 0$ we consider $X \restriction s$, and for each k we define $t_{k,s}$ to be the largest stage $t \leq s$ such that $X \restriction s \in \mathcal{O}_{k,t}$. (Let $t_{k,s} = 0$ if such a t does not exist.) Define k to be *expansionary* at stage s if $t_{k,s-1} < t_{k,s}$. Now we let $k(s) = \min\{k : k \text{ is expansionary at } s\}$. There are two substages.

Substage (a). First consider all strings σ extending Y_{s-1} of minimal length with $\mathbf{d}(\sigma) \geq 2^{\frac{1}{k(s)}|\sigma|}$, and take the leftmost one of these σ's. Such σ's exist because $S^{2^{\frac{1}{k(s)}n}}[\mathbf{d}]$ is dense. If $k(s)$ does not exist, let $\sigma = Y_{s-1}$.

Substage (b). Next consider all extensions $\tau \sqsupseteq \sigma$ of minimal length such that $\mathbf{d}(\tau \restriction i) \leq \mathbf{d}(\tau \restriction (i-1))$ for every $|\sigma| < i < |\tau|$, and $\mathbf{d}(\tau) \leq |\tau|$. Clearly such τ exist, by direct diagonalization against $\mathbf{d}$. Define Y_s to be the leftmost of these τ. This concludes the construction.

So Y_s is defined by first building a piece of evidence σ that $\mathbf{d}$ achieves growth rate $2^{\frac{1}{k(s)}n}$ on Y and then slowing down the growth rate of $\mathbf{d}$ to the order n. Note that f is continuous. If $X \in \bigcup_k \bigcap_s \mathcal{O}_{k,s}$, then for the minimal k such that $X \in \bigcap_s \mathcal{O}_{k,s}$, infinitely many pieces of evidence σ

witness that $\mathbf{d}$ achieves growth rate $2^{\frac{1}{k}n}$ on Y, so $Y \notin \mathrm{DIM}^1$. On the other hand, if $X \notin \bigcup_k \bigcap_s \mathcal{O}_{k,s}$ then for every k only finitely often $\mathbf{d}(Y_s) \geq 2^{\frac{1}{k}|Y_s|}$ because in substage (a) the extension σ is chosen to be of minimal length, so $Y \notin S_{h_k}[\mathbf{d}]$. Hence $Y \in \mathrm{DIM}^1$. □

As RAND is a Σ^0_2-class, we have the following corollary (which can also be proved by a direct construction).

Corollary 4.10. (Lutz [14]) RAND *is a proper subset of* DIM^1.

In order to establish the existence of Δ^0_2-computable sequences of any Δ^0_2-computable dimension $\alpha \in [0,1)$, Lutz [14] gave a *dilution function* $g_\alpha : \mathbf{C} \to \mathbf{C}$ that is computable and satisfies $\dim(g_\alpha(X)) = \alpha \cdot \dim(X)$ for all $X \in \mathbf{C}$. Applying this to any Δ^0_2-computable Martin-Löf random sequence (which must have dimension 1) establishes the existence theorem. As g_α is continuous, it is a Wadge reduction from DIM^1 to DIM^α if $\alpha > 0$. Combining this with the previous theorem, we have that DIM^α is Wadge complete for $\mathbf{\Pi^0_3}$ for all Δ^0_2-computable $\alpha \in (0,1)$. We now give a similar dilution construction that will allow us to prove this for *arbitrary* $\alpha \in (0,1)$.

Let $X \in \mathbf{C}$ and let $\alpha \in (0,1)$. Write $X = x_1 x_2 x_3 \ldots$ where $|x_n| = 2n-1$ for all n, noting that $|x_1 \cdots x_n| = n^2$. For each n, let

$$k_n = \left\lceil n \frac{1-\alpha}{\alpha} \right\rceil$$

and $y_n = 0^{k_n}$. We then define

$$f_\alpha(X) = x_1 y_1 x_2 y_2 \cdots x_n y_n \cdots \ .$$

Observe that f_α is a continuous function mapping $\mathbf{C}$ to $\mathbf{C}$. We now show that it modifies the dimension of X in a controlled manner.

Lemma 4.11. *For any* $X \in \mathbf{C}$ *and* $\alpha \in (0,1)$,

$$\dim(f_\alpha(X)) = \alpha \cdot \dim(X)$$

and

$$\mathrm{Dim}(f_\alpha(X)) = \alpha \cdot \mathrm{Dim}(X).$$

Proof. The proof uses (2.5) and (2.6), the Kolmogorov complexity characterizations of dimension and strong dimension.

Let $w \sqsubseteq f_\alpha(X)$. For some n,

$$w = x_1 y_1 \cdots x_{n-1} y_{n-1} v,$$

where $v \sqsubseteq x_n y_n$. Then

$$\begin{aligned} K(w) &\leq K(x_1 \cdots x_{n-1}) + K(v) \\ &\quad + K(k_1) + \cdots + K(k_{n-1}) + O(1) \\ &\leq K(x_1 \cdots x_{n-1}) + O(n \log n). \end{aligned}$$

Because

$$|w| \geq |x_1 y_1 \cdots x_{n-1} y_{n-1}| \geq \frac{(n-1)^2}{\alpha},$$

we have

$$\frac{K(w)}{|w|} \leq \frac{\alpha \cdot K(x_1 \cdots x_{n-1})}{|x_1 \cdots x_{n-1}|} + \frac{O(n \log n)}{(n-1)^2},$$

It follows that

$$\begin{aligned} \dim(f_\alpha(X)) &\leq \alpha \liminf_{n\to\infty} \frac{K(x_1 \cdots x_{n-1})}{|x_1 \cdots x_{n-1}|} \\ &= \alpha \liminf_{n\to\infty} \frac{K(x\restriction n)}{n} \\ &= \alpha \cdot \dim(X), \end{aligned}$$

where the first equality holds because the blocks x_n are short relative to $x_1 \cdots x_{n-1}$. Similarly, $\mathrm{Dim}(f_\alpha(X)) \leq \alpha \cdot \mathrm{Dim}(X)$.

For the other inequality, we have

$$\begin{aligned} K(x_1 \cdots x_{n-1}) &\leq K(w) + K(k_1) + \cdots + K(k_{n-1}) \\ &\quad +O(1) \\ &\leq K(w) + O(n \log n) \end{aligned}$$

and

$$|w| \leq |x_1 y_1 \cdots x_n y_n| \leq \frac{n^2}{\alpha} + n \leq \frac{(n+1)^2}{\alpha},$$

so

$$\begin{aligned} \frac{K(w)}{|w|} &\geq \alpha \frac{K(x_1 \cdots x_{n-1}) - O(n \log n)}{(n+1)^2} \\ &= \alpha \frac{K(x_1 \cdots x_{n-1})}{|x_1 \cdots x_{n-1}|} \frac{(n-1)^2}{(n+1)^2} - \frac{O(n \log n)}{(n+1)^2} \end{aligned}$$

Therefore

$$\begin{aligned} \dim(f_\alpha(X)) &\geq \alpha \liminf_{n\to\infty} \frac{K(x_1 \cdots x_{n-1})}{|x_1 \cdots x_{n-1}|} \\ &= \alpha \liminf_{n\to\infty} \frac{K(x\restriction n)}{n} \\ &= \alpha \cdot \dim(X), \end{aligned}$$

and analogously, $\mathrm{Dim}(f_\alpha(X)) \geq \alpha \cdot \mathrm{Dim}(X)$. □

The function f_α establishes the completeness of DIM^α.

Theorem 4.12. *For all $\alpha \in (0,1)$, the DIM^α is Wadge complete for $\boldsymbol{\Pi}^0_3$. Therefore it is not a $\boldsymbol{\Sigma}^0_3$-class, and in particular not a Σ^0_3-class.*

Proof. By Lemma 4.11, f_α is a Wadge reduction from DIM^1 to DIM^α. Therefore DIM^α is Wadge complete for $\boldsymbol{\Pi}^0_3$ by composing f_α with the reduction from Theorem 4.9. □

As g_α is also a Wadge reduction from $\mathrm{DIM}^1_{\mathrm{str}}$ to $\mathrm{DIM}^\alpha_{\mathrm{str}}$, we have from Theorem 4.5 that $\mathrm{DIM}^\alpha_{\mathrm{str}}$ is not a $\boldsymbol{\Sigma}^0_2$-class for all $\alpha \in (0,1)$. We now prove that $\mathrm{DIM}^\alpha_{\mathrm{str}}$ is not even $\boldsymbol{\Sigma}^0_3$ for all $\alpha \in [0,1)$.

Theorem 4.13. *For all $\alpha \in [0,1)$, $\mathrm{DIM}^{\alpha}_{\mathrm{str}}$ is Wadge complete for $\mathbf{\Pi^0_3}$. Therefore $\mathrm{DIM}^{\alpha}_{\mathrm{str}}$ is not a $\mathbf{\Sigma^0_3}$-class, and in particular is not a $\Sigma^0_3[\Pi^0_1]$-class.*

Proof. The proof is similar to that of Theorem 4.9, but uses (2.6), the Kolmogorov complexity characterization of strong dimension. Let $\mathcal{C} = \bigcup_k \bigcap_s \mathcal{O}_{k,s}$ be a $\mathbf{\Sigma^0_3}$-class and without loss of generality assume that $\mathcal{O}_{k,s} \supseteq \mathcal{O}_{k,s+1}$ for all k,s.

Let $\alpha \in (0,1)$. (We will discuss the simpler case $\alpha = 0$ later.) We define a continuous function $f : \mathbf{C} \to \mathbf{C}$ in stages that will Wadge reduce $\mathcal{C}$ to $\overline{\mathrm{DIM}^{\alpha}_{\mathrm{str}}}$. The image $Y = f(X)$ will be the unique sequence extending Y_s for all s. At stage 0 we define Y_0 to be the empty sequence.

At stage $s > 0$ we consider $X{\upharpoonright}s$, and define $k(s)$ as in the proof Theorem 4.9. There are three substages.

Substage (a). First consider all strings ρ extending Y_{s-1} of minimal length with $K(\rho) \geq \alpha|\rho|$, and take the leftmost one of these ρ's.

Substage (b). Next consider all strings σ extending ρ of minimal length with $K(\sigma) \geq (\alpha + \frac{1}{k(s)})|\sigma|$, and take the leftmost one of these σ's. If $k(s)$ does not exist, let $\sigma = \rho$.

Substage (c). Extend σ with a block of 0's to obtain $Y_s = \sigma 0^{|\sigma|^2 - |\sigma|}$.

That is, to define Y_s, we first select ρ to increase the Kolmogorov complexity rate to α. This ensures that Y will have strong dimension at least α. We then construct a piece of evidence σ that Y has strong dimension at least $\alpha + \frac{1}{k(s)}$. We finish Y_s with a long block of 0's to bring the Kolmogorov complexity down to a near-zero rate, so that the next stage will work properly.

If $X \in \mathcal{C}$, then for the minimal k such that $X \in \bigcap_s \mathcal{O}_{k,s}$, infinitely many prefixes $\sigma \sqsubseteq Y$ satisfy $K(\sigma) \geq (\alpha + \frac{1}{k})|\sigma|$. Therefore $\mathrm{Dim}(Y) \geq \alpha + \frac{1}{k}$, so $Y \notin \mathrm{DIM}^{\alpha}_{\mathrm{str}}$.

Now let $X \notin \mathcal{C}$. Let $\alpha' > \alpha$ be arbitrary, and choose k so that $\frac{1}{k} < \alpha' - \alpha$. Because $X \notin \mathcal{C}$, we have $k(s) > k$ for all sufficiently large s. Let s_0 be large enough to ensure $k(s) > s$ and $K(Y_{s-1}) \leq \sqrt{|Y_{s-1}|} + O(1) < \alpha|Y_{s-1}|$ hold for all $s \geq s_0$. Suppose that

$$K(w) \geq \alpha'|w|. \tag{4.2}$$

holds for some w with $Y_{s-1} \sqsubseteq w \sqsubseteq Y_s$ for some stage $s \geq s_0$. We then have that ρ is a proper extension of Y_{s-1}. By choice of ρ and σ and the fact that $\alpha' > \alpha + \frac{1}{k} > \alpha + \frac{1}{k(s)}$, we must have $w = \rho$ or $\sigma \sqsubseteq w$. We analyze these two cases separately.

(i) $w = \rho$: Let ρ' be the string obtained from ρ by removing the last bit. Then $K(\rho) \leq K(\rho') + O(1)$. By choice of ρ, we have $K(\rho') < \alpha|\rho'|$. We also have $K(\rho) \geq (\alpha')|\rho|$ by (4.2). Putting these three statements together yields

$$\alpha'|\rho| < \alpha(|\rho| - 1) + O(1),$$

which is a contradiction if $|\rho| = |w|$ is sufficiently large.

(ii) $\sigma \sqsubseteq w$: Obtain σ' from σ by removing the last bit of σ. Then we have

$$\begin{aligned} K(w) &\leq K(\sigma') + K(|w| - |\sigma|) + O(1) \\ &\leq K(\sigma') + \log(|w| - |\sigma|) + O(1) \\ &\leq K(\sigma') + 2\log|\sigma| + O(1). \end{aligned}$$

By choice of σ, $K(\sigma') < (\alpha + \frac{1}{k(s)})|\sigma'|$. These two facts together with (4.2) tell us that

$$\alpha'|w| < \left(\alpha + \frac{1}{k(s)}\right)(|\sigma| - 1) + 2\log|\sigma| + O(1),$$

which is a contradiction for large $|w|$ because $|w| \geq |\sigma|$ and $\alpha' > \alpha + \frac{1}{k(s)}$.

Therefore, for all sufficiently long $w \sqsubseteq Y$, (4.2) does not hold. It follows that $\mathrm{Dim}(Y) \leq \alpha$. On the other hand, there are infinitely many $\rho \sqsubseteq Y$ with $K(\rho) \geq \alpha|\rho|$, so $\mathrm{Dim}(Y) \geq \alpha$. Therefore $Y \in \mathrm{DIM}^\alpha_{\mathrm{str}}$.

This shows that f is a Wadge reduction from $\mathcal{C}$ to $\overline{\mathrm{DIM}^\alpha_{\mathrm{str}}}$. As $\mathcal{C}$ is an arbitrary $\mathbf{\Sigma^0_3}$-class, this shows that $\mathrm{DIM}^\alpha_{\mathrm{str}}$ is Wadge complete for $\mathbf{\Pi^0_3}$.

The proof for the case $\alpha = 0$ is similar, but simpler as substage (a) is omitted in the construction.

□

We conclude this section by summarizing its main results in the following table.

	DIM^α	$\mathrm{DIM}^\alpha_{\mathrm{str}}$
$\alpha = 0$	$\Pi^0_2 - \mathbf{\Sigma^0_2}$	$\Pi^0_3[\Sigma^0_1] - \mathbf{\Sigma^0_3}$
$\alpha \in (0,1) \cap \Delta^0_2$	$\Pi^0_3 - \mathbf{\Sigma^0_3}$	$\Pi^0_3[\Sigma^0_1] - \mathbf{\Sigma^0_3}$
$\alpha = 1$	$\Pi^0_3 - \mathbf{\Sigma^0_3}$	$\Pi^0_2[\Pi^0_1] - \mathbf{\Sigma^0_2}$
arbitrary $\alpha \in (0,1)$	$\mathbf{\Pi^0_3} - \mathbf{\Sigma^0_3}$	$\mathbf{\Pi^0_3} - \mathbf{\Sigma^0_3}$

5 Effective Randomness Classes

We begin this section by pointing out some relationships between computable dimension, Church randomness, and Schnorr randomness.

Analogously to what was done for the constructive case, the *computable dimension* of a class $\mathcal{A} \subseteq \mathbf{C}$ is defined as

$$\dim_{\mathrm{comp}}(\mathcal{A}) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there exists a computable} \\ s\text{-gale } d \text{ for which } \mathcal{A} \subseteq S^\infty[d] \end{array} \right\}.$$

A *selection rule* is a function $\varphi : \{0,1\}^* \to \{0,1\}$. With every selection rule φ we associate a function $\Phi : \{0,1\}^* \to \{0,1\}^*$ defined by $\Phi(\lambda) = \lambda$ and

$$\Phi(wi) = \begin{cases} \Phi(w)i & \text{if } \varphi(w) = 1, \\ \Phi(w) & \text{if } \varphi(w) = 0. \end{cases}$$

A set A is called *Church random* if every substring of χ_A (the characteristic string of A) defined by a computable selection rule is stochastic, i.e., satisfies the law of large numbers. Consider the following property of selection rules:

$$\inf_{w \in \{0,1\}^*} \frac{|\Phi(w)|}{|w|} > 0. \tag{5.1}$$

A *computable null set of exponential order* is a set of the form $S^{a^n}[d]$, where d is a computable martingale and $a > 1$. It is easy to check that a set is not in any computable null set of exponential order if and only if $\{A\}$ has computable dimension 1. With this observation, we can restate a result of Schnorr as follows.

Theorem 5.1. (Schnorr [19, Satz 17.8]) *$\{A\}$ has computable dimension 1 if and only if every substring of χ_A selected by a computable selection rule with the property (5.1) is stochastic.*

In particular every Church random sequence is not in any null set of the form $S^{a^n}[d]$ where d is computable. In the words of Schnorr [20], "Church random sequences approximate the behavior of Schnorr random sequences."

Proposition 5.2. *There are sequences with computable dimension 1 that are not Church random.*

Proof. Let R be computably random, and let $D = \{2^n \mid n \in \mathbb{N}\}$ be an exponentially sparse decidable domain. Then $A = R - D$ has computable dimension 1, but D can be computably selected, so A is not Church random. $\square$

We now classify the Schnorr random sequences in the arithmetical hierarchy.

Theorem 5.3. $\mathrm{RAND}_{\mathrm{Schnorr}}$ *is a* Π^0_3*-class, but not a* $\mathbf{\Sigma^0_3}$*-class.*

Proof. First note that $\mathrm{RAND}_{\mathrm{Schnorr}} \in \Pi^0_3$: $A \in \mathrm{RAND}_{\mathrm{Schnorr}}$ if and only if for every pair of codes e and f, either the e-th partial computable function φ_e is not a computable order (i.e. is not total or decreases at some point), φ_f is not a computable martingale (i.e. is not total or violates the martingale property at some point), or $A \notin S^{\varphi_e}[\varphi_f]$, and that every one of these options is Σ^0_2.

The rest of the proof resembles that of Theorem 4.9. Fix a (non-computable) sequence of computable martingales $\{d_k\}_{k\in\mathbb{N}}$ and a sequence of computable orders $\{h_k\}_{k\in\mathbb{N}}$ such that

(i) $A \in \mathrm{RAND}_{\mathrm{Schnorr}} \iff \forall k (A \notin S^{h_k}[d_k])$.

(ii) $S^{h_k}[d_k] - S^{\min\{h_j : j<k\}}[\sum_{j<k} d_j]$ is dense for every k.

The d_k can be defined by taking appropriate sums of computable martingales so that for any computable martingale d, there is some d_k such that $d_k(w) \geq d(w)$ for all w. For the h_k one can take any family of computable orders such that every computable order h dominates some h_k. (Of course the d_k and h_k cannot be uniformly computable families, but that is of no concern to us.)

Let $\bigcup_k \bigcap_s \mathcal{O}_{k,s}$ be a $\mathbf{\Sigma^0_3}$-class. We define a continuous function $f : \mathbf{C} \to \mathbf{C}$ such that

$$\forall k \left(X \in \bigcap_s \mathcal{O}_{k,s} \iff f(X) \in S^{h_k}[d_k] \right) \tag{5.2}$$

so that by (i) we have $X \notin \bigcup_k \bigcap_s \mathcal{O}_{k,s} \iff f(X) \in \mathrm{RAND}_{\mathrm{Schnorr}}$.

As in the proof of Theorem 4.9 we define the image $Y = f(X)$ in stages. Every time we find a new piece of evidence that $X \in \bigcap_s \mathcal{O}_{k,s}$, at stage s say, we build a piece of evidence that $Y \in S^{h_k}[d_k]$ by choosing an appropriate finite extension at stage s. Such an extension can be found by (ii). The rest of the proof is identical to that of Theorem 4.9. $\square$

With only some obvious changes one can also prove the following theorem.

Theorem 5.4. $\mathrm{RAND}_{\mathrm{comp}}$ *is a* Π^0_3*-class, but not a* $\mathbf{\Sigma^0_3}$*-class.*

Proof. Note that X is computably random if and only if for every e, φ_e is not a computable martingale or $X \notin S^\infty[\varphi_e]$, so the class is Π^0_3. That it is properly Π^0_3 is actually easier than the proof of Theorem 5.3 since we only need the sequence $\{d_k\}$ and not the $\{h_k\}$. $\square$

In contrast to the universal constructive supermartingale $\mathbf{d}$ satisfying $\mathrm{RAND} = \overline{S^\infty[\mathbf{d}]}$, Theorem 5.4 implies that, even from a noncomputable standpoint, $\mathrm{RAND}_{\mathrm{comp}}$ has no such universal object. That is, $\mathrm{RAND}_{\mathrm{comp}} \neq \overline{S^\infty[d]}$ for any (arbitrarily noncomputable) supermartingale d, as otherwise $\mathrm{RAND}_{\mathrm{comp}}$ would be a $\mathbf{\Sigma^0_2}$-class.

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